# Hermite Interpolation with Mathematica

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# Problem

Given three finite lists, X, F, DF.

$X = \{x_1, x_2, \ldots, x_n\}$ wlog $(x_1 < x_2 < \ldots < x_n)$
$F = \{f_1, f_2, \ldots, f_n\}$
$\mathrm{DF} = \{\mathrm{df}_1, \mathrm{df}_2, \ldots, \mathrm{df}_n\}$ $(x_j, f_j, \mathrm{df}_j \in \mathbb{R})$.

Find a poly H such that $\forall_{1 \le j \le n}\ H[x_j] = f_j \bigwedge H'[x_j] = \mathrm{df}_j$,

## Theorem

$\exists!\ H$ of degree at most $2n - 1$ which satisfies all the interpolation constraints.

Question: What is the Newtonian Form of $H$?

## Example

$X = \{1, 3\}, \; F = \{2, 0\}, \; \mathrm{DF} = \{0, 1\}$

```
pe[x_] = 2 - 1/2 (x - 1)^2 + 3/4 (x - 3) (x - 1)^2;
```

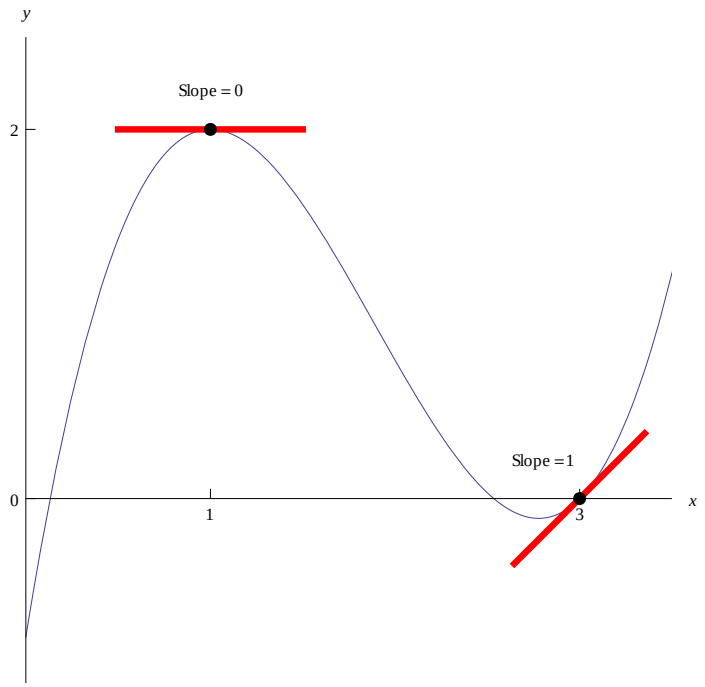
```
Expand[pe[x]]
```

$$-\frac{3}{4} + \frac{25\,x}{4} - \frac{17\,x^2}{4} + \frac{3\,x^3}{4}$$

```
{{pe[1], pe[3]}, {D[pe[x], x] /. x → 1, D[pe[x], x] /. x → 3}} // TableForm
```

| | |
|---|---|
| 2 | 0 |
| 0 | 1 |

## Example



## Remark (Uniqueness)

```
pe2[x_] = 2 - 1/2 (x - 1)^2 + 3/4 (x - 1)^2 (x - 3) + (x - 1)^2 (x - 3)^2;
{{pe2[1], pe2[3]}, {D[pe2[x], x] /. x → 1, D[pe2[x], x] /. x → 3}} // TableForm
```

| | |
|---|---|
| 2 | 0 |
| 0 | 1 |

## Algorithm Poly-Seed

Idea: Construct the poly H step by step considering only one extra interpolation constraint per round

Observation: We can put the interpolation constraints in several different order, the only restriction is: at any point never use the constraint on the derivative $(\mathrm{df}_j)$ before the constraint on the function value $(f_j)$

# Example

Init. poly of degree 0

```
p0 = F[[1]]
```

2

```
p1 = p0 + A (x - 1);
```

```
Solve[(D[p1, x] /. x → X[[1]]) == DF[[1]]]
```

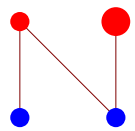
{{A → 0}}

$p1 = p0 + 0\,(x - x1)$,
$p2 = p1 + A\,(x - x1)^2$ etc.,

Apriori Form, Sucessive Solving of Linear Equations to Get Coeffs A.

$p0,\ p1,\ p2,\ \ldots,\ p_{2\,n-1} = H$

# Intapp: Newtonian Forms of the Hermite Polynomial

| | | |
|---|---|---|
| DF: 0 1<br>F: 2 0 |  | ⇑ ⇓ ⇐ ⇒ + I |
| x | 1 3 | $2 - \frac{1}{2}(-1+x)^2 + \frac{3}{4}(-3+x)(-1+x)^2$ |

n

Hermite Interpolation 2012

# Coeffs (2,0,-1/2,3/4)

Classical order, generalized Newton Scheme

| | | | | |
|---|---|---|---|---|
| 1 | 2 | | | |
| 1 | 2 | 0 | | |
| 3 | 0 | -1 | $-\frac{1}{2}$ | |
| 3 | 0 | 1 | 1 | $\frac{3}{4}$ |

## Coeffs (2,0,-1/2,3/4) and Series Expansion [Closed Form]

H=∑j A[j] ω[j], where A[j]=∑k y[k] Γn,k

How do we get e.g. the lc=3/4? Compute the Γ's as coeffs of series expansions:

```
scf3 =
 {SeriesCoefficient[1/(x - X[[2]])^2, {x, X[[1]], 1}],
  SeriesCoefficient[1/(x - X[[2]])^2, {x, X[[1]], 0}], SeriesCoefficient[
   1/(x - X[[1]])^2, {x, X[[2]], 1}], SeriesCoefficient[1/(x - X[[1]])^2, {x, X[[2]], 0}]}
```

$$\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right\}$$

```
A[3] = {F[[1]], DF[[1]], F[[2]], DF[[2]]}.scf3
```

$$\frac{3}{4}$$

## Coeffs (2,0,-1/2,3/4) and Series Expansion [Closed Form]

H=∑j A[j] ω[j], where A[j]=∑k y[k] Γn,k

How do we get e.g. the 2nd c=-1/2? Compute the Γ's as coeffs of series expansions:

```
scf2 =
 {SeriesCoefficient[1/(x - X[[2]])^1, {x, X[[1]], 1}],
  SeriesCoefficient[1/(x - X[[2]])^1, {x, X[[1]], 0}],
  SeriesCoefficient[1/(x - X[[1]])^2, {x, X[[2]], 0}]}
```

$$\left\{-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right\}$$

```
A[2] = {F[[1]], DF[[1]], F[[2]]}.scf2
```

$$-\frac{1}{2}$$

## Outlook

– more general problem setting: constraints on higher order derivatives
– interpolation over finite fields (poly↔poly function)
– numerical issues, computational complexity
– applicatons: Triangular decompositions of Hermite-Vandermonde matrices

## Literature

This talk is based on the paper

L. L. Stacho — R. Vajda:
Hermite interpolation sequences over finite fields (2012, preprint, submitted)

# Aux

## Hermite aux