

***Functions, Dynamic  
Geometry and CAS:  
offering possibilities for  
learners and teachers***

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# *Functions, DG and CAS*

- The CaSyOPEE research group
- What can we learn from math education research about functions?
- How can we use it to implement situations of use of CAS and DG?

# *The CaSyOPEE research group*

- 1995-2000 DERIVE, the TI-92.  
Instrumental Approach
- 2000-2006 Transposing CAS  
Building a CAS tool for classroom
- 2006-2010 The ReMath project  
Focus on multi-representation
- 2010-... Dissemination to teachers.
  - usable tool
  - conceptual framework about functions and algebra



# *The Casyopee research group*

1995-2000  
DERIVE, the  
TI-92.

Instrumental  
Approach



*Complex calculators in  
the classroom:  
theoretical and  
practical reflections on  
teaching pre-calculus*

INTERNATIONAL  
JOURNAL OF  
COMPUTERS FOR  
MATHEMATICAL  
LEARNING

Volume 4, Number 1  
(1999)

- Tasks and techniques to help students to develop an appropriate instrumental genesis for algebra and functions

- Potential of the calculator for connecting enactive representations and theoretical calculus

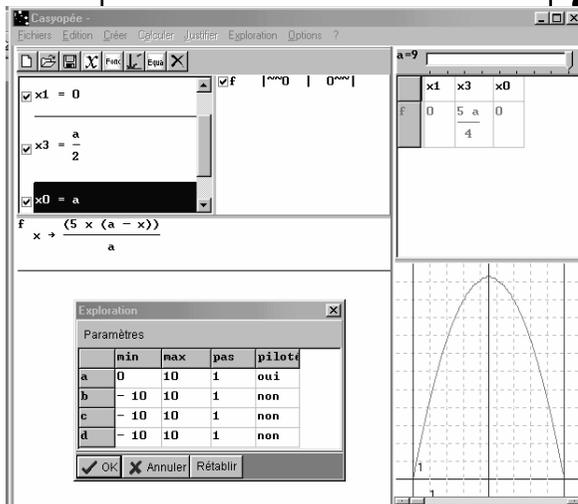
# The Casyopee research group

2000-2006  
Transposing  
CAS  
Building a tool  
for the  
classroom

Curriculum,  
classroom practices,  
and tool design in  
the learning of  
functions through  
technology-aided  
experimental  
approaches

- while symbolic calculation is a basic tool for mathematicians, curricula and teachers are very cautious
- design and experiment of a computer environment as means to contribute to an evolution of curricula and classroom practices

*International journal of  
computers for  
mathematical learning  
Volume 10, Number 2  
(2005)*



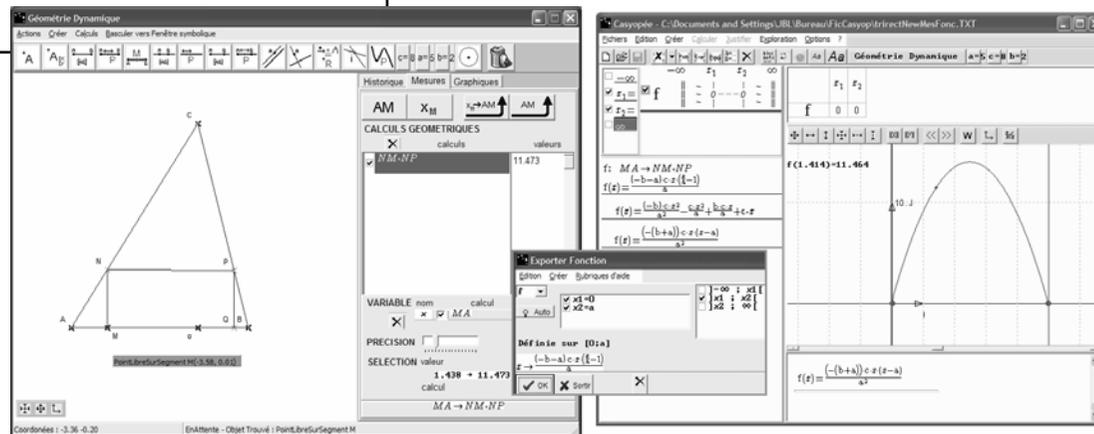
# The Casyopee research group

2006-2010  
The ReMath  
project

Focus on  
multi-  
representa-  
tion

Teaching and learning about  
functions at upper secondary  
level: designing and  
experimenting the software  
environment Casyopée.  
*International Journal of  
Mathematical Education in  
Science and Technology*  
Volume 41, Issue 2, 2010

An experimental  
teaching unit carried  
out in the ReMath  
European project  
focusing on the  
approach to functions  
via multiple  
representations for the  
11th grade.



# The Casyopee research group

2010-... Dissemination to teachers.

-usable tool

-conceptual framework about functions and algebra

Students' activities about functions at upper secondary level  
*33rd PME Conf. July 2009*

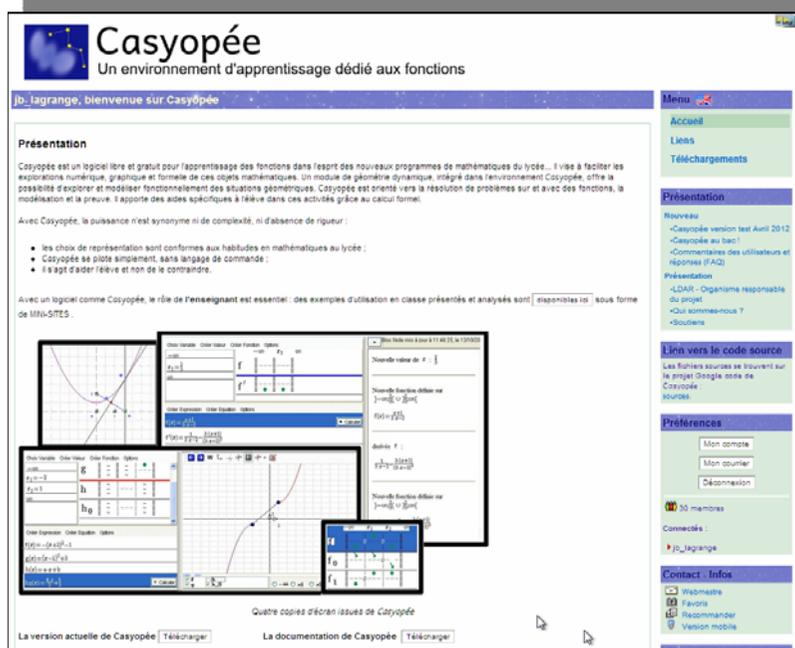
Working with

teachers:

Innovative software at the boundary between research and classroom.  
*35th PME Conf. July 2001*

•A grid that organises and connects various students' activities about functions at upper secondary level.

•Diffusion of research outcomes through communal work involving researchers and teachers



- What can we learn from math education research about functions?
  - A functional perspective on the teaching of algebra (Kieran)
  - Some key ideas (Lagrange Psycharis)

# *A functional perspective on the teaching of algebra (Kieran)*

- Quite **widespread** in the world
- Attempt to include elements of **both traditional** (rational expressions and equations) **and functional orientations** to school algebra.
- The orientation toward the solving of realistic problems, with the **aid of technological tools**, allows for an algebraic content that is **less manipulation oriented**.
- Such orientations also emphasize **multirepresentational** activity with a **shift away from the traditional skills** of algebra

# *A functional perspective on the teaching of algebra (Kieran)*

## **Objections**

1. Expressions can be used **without describing functions**, and functions can be expressed without using algebra
2. Students become confused **regarding distinctions between equations and functions**, not being able to sort out, for example, how equivalence of equations is different from equivalence of functions.
3. There is a strong presumption that **symbolic forms are to be interpreted graphically**, rather than dealt with, **technology being used to insist on screen (graphical) interpretations** of functions  
→ no meaning given to **symbolic forms**.

# Teaching/learning about functions : **some key ideas**

- From **process-object** to **co-variation** aspect of functions
- The role of **symbolism**
- Understanding **co-variation**
- Understanding **independent variable**

# From process-object ...

- early nineties : distinction between
  - **process view** characterised by students' focus on the **performance of computational actions** following a sequence of operations (i.e. computing values)
  - **object view** based on the **generalization of the dependency relationships** between input-output pairs of two quantities/magnitudes

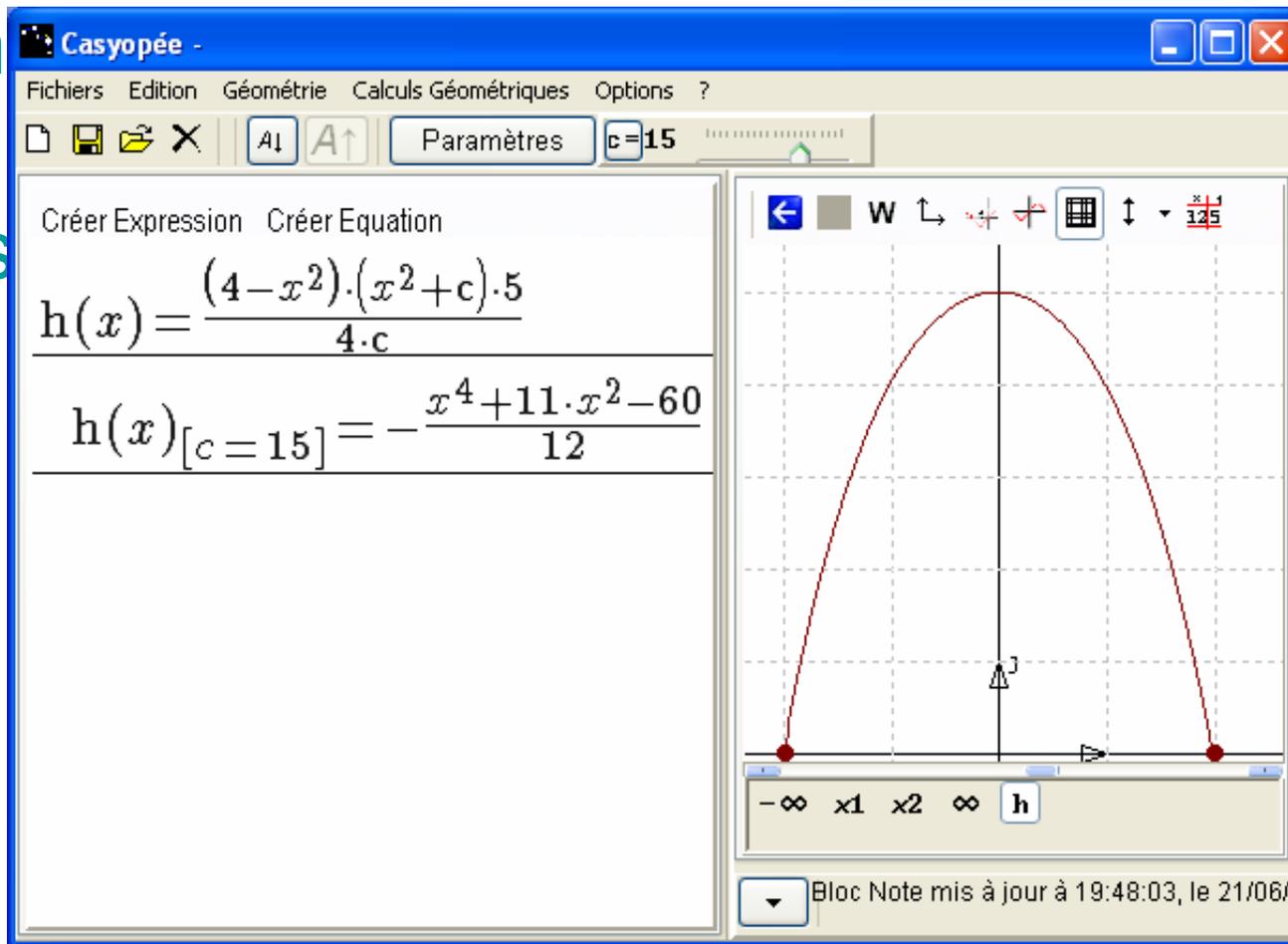
## ... to co-variation

- **Object-oriented** views of function emphasize the co-variation aspect of function
- The **co-variation** view of functions
  - understanding the manner in which dependent and independent variables change as well as the coordination between these changes.
- A shift in understanding an expression  $f(x) = 2^x$ 
  - from a **single input-output view**,
  - to a more dynamic way : ‘**running through**’ a **continuum of numbers**’
- Not obvious for the students

# The role of symbolism

- Critical role of symbolism in the development of the function concept.
  - “confronted in very different forms such as graphs and equations”
- Need for students’ investigation of algebraic and functional ideas in different contexts such as the geometric one.

- Critical role of symbolism in the development of the function concept.
- “confronted in very different forms such as graphs and equations”



- Need for students' investigation of algebraic and functional ideas in different contexts such as the geometric one.

The screenshot shows the Casyopée software interface. On the left, a geometric diagram features a right-angled triangle with vertices N (top), Q (bottom-left), and P (bottom-right). A horizontal line segment MP is drawn from vertex M on the vertical side NQ to vertex P on the horizontal side QP. The area under the line NP and above the line MP is shaded green. The software's toolbar includes various geometric construction tools like points, lines, and circles. The main workspace on the right contains algebraic expressions:

$$c0 = \frac{(NQ + MP) \cdot QP}{2}$$

$$c1 = QP$$

Below these, a function definition is shown:

$$f: QP \rightarrow \frac{(NQ + MP) \cdot QP}{2}$$

$$f(x) = \frac{x^2 + 4 \cdot x}{2}$$

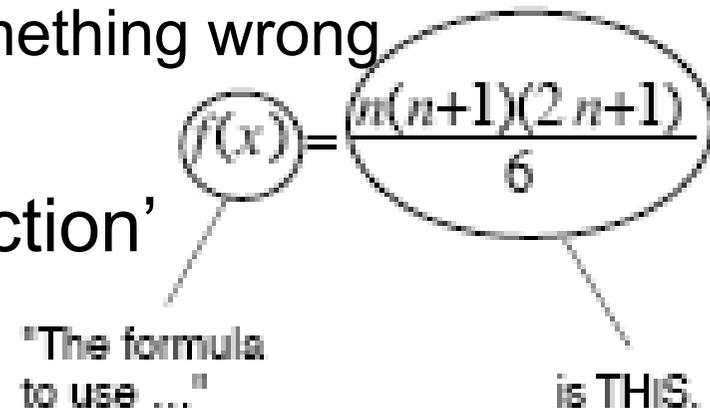
The status bar at the bottom indicates the coordinates as 1.20 0.12 and shows a timestamp: 'Bloc Note mis à jour à 20:03:24, le 21/06/2012.'

# Understanding **co-variation**

- Built upon understanding of correspondence in a **very long term process**.
- Situations based on **modeling dynamic phenomena**
  - connect functions with a **sensual experience of dependency**
  - have a potential to **help students reach awareness of co-variation**.
  - can help students to gradually **understand the properties of functions by connecting in a meaningful way its different representations**

# Understanding **independent variable**

- Students' persistent **'mal-formed concept images** showing up in the strangest places".
- An example: the formula for the sum of first  $n$  squares
  - A student wrote  $f(x) = n(n+1)(2n+1)/6$
  - none of the students found something wrong
- Predominant image evoked in students' by the word 'function' **two disconnected expressions linked by the equal sign.**

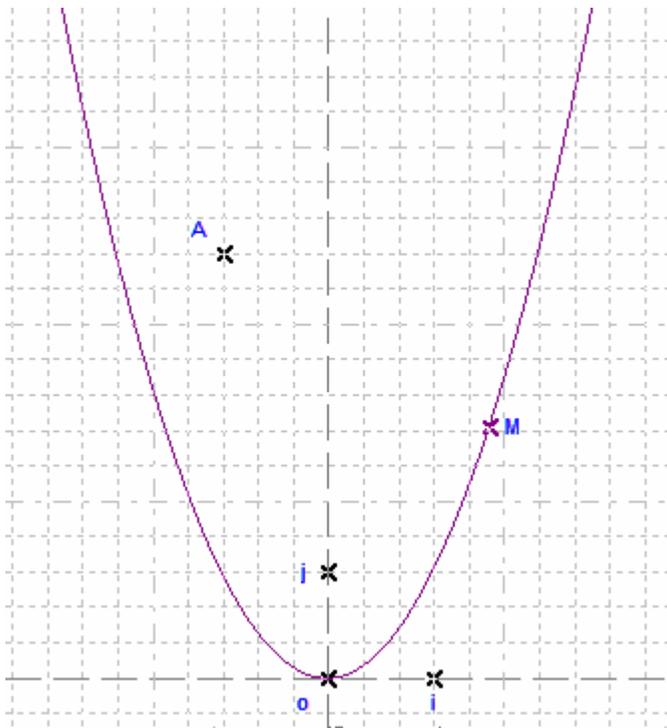


How can we use these ideas to implement situations of use of CAS and DG?

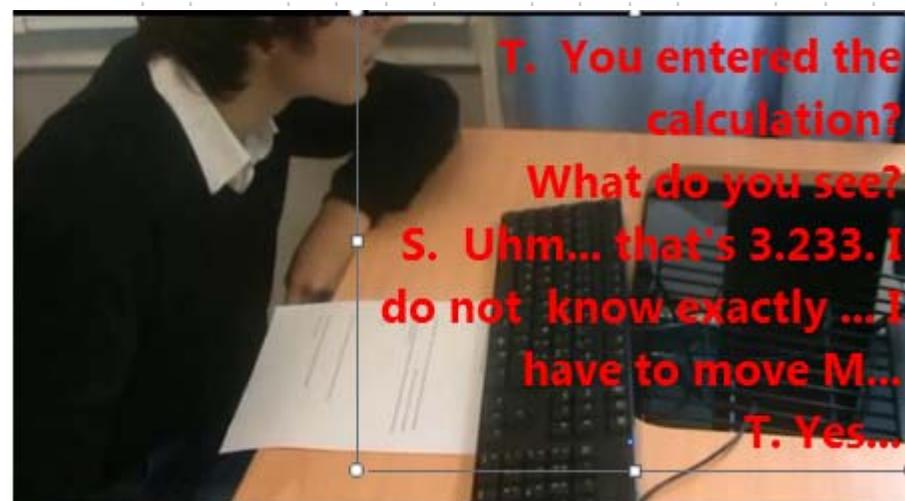
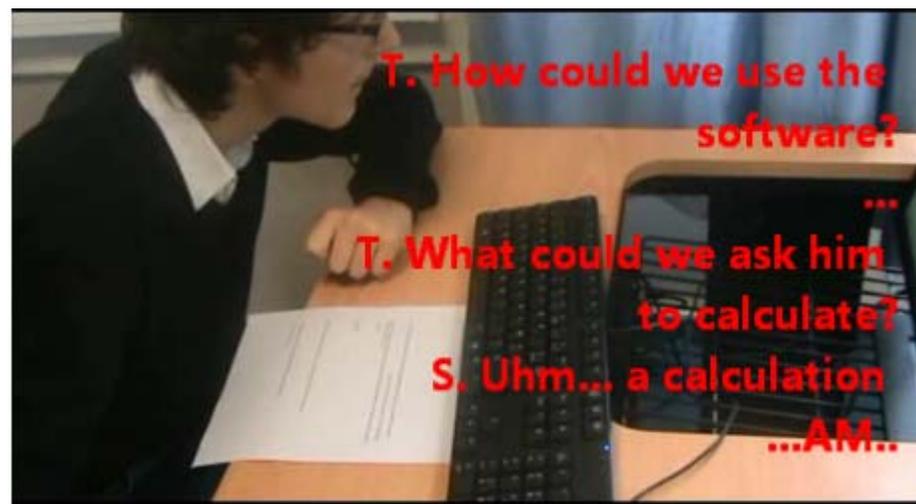
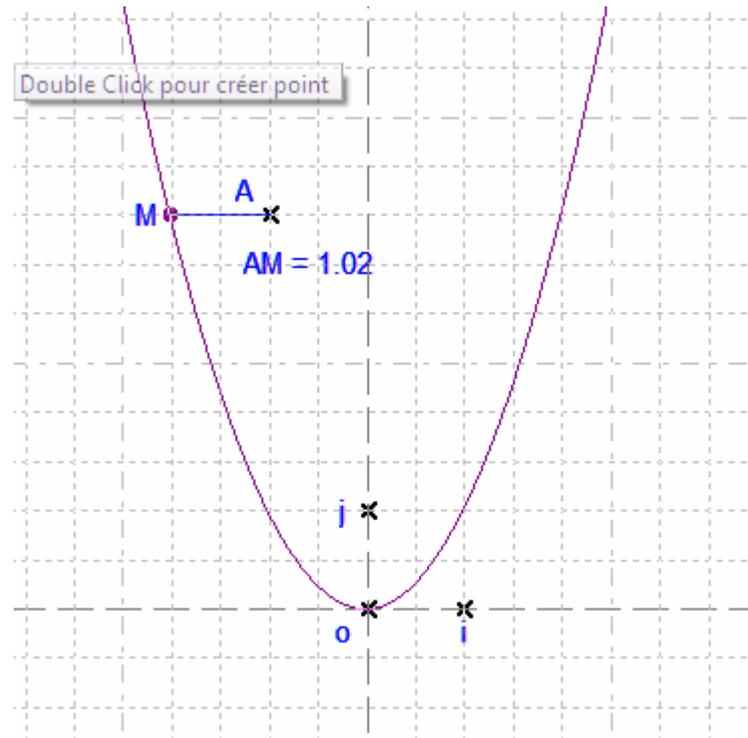
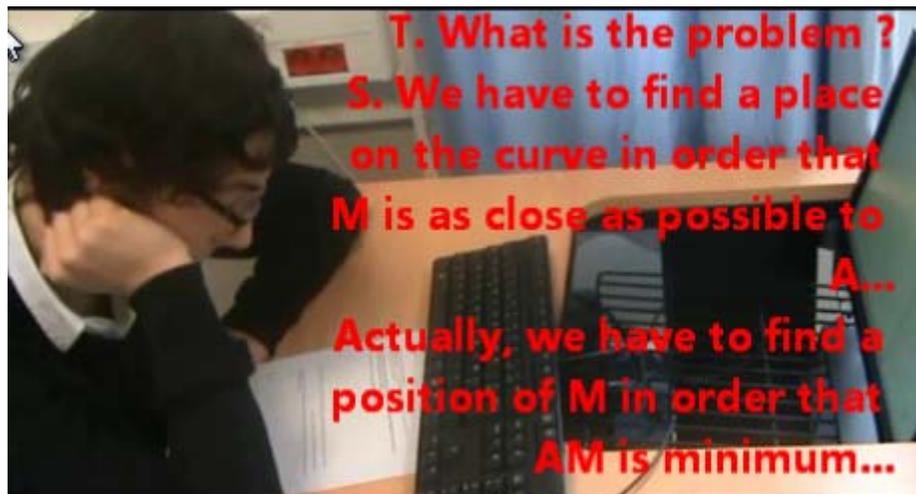
## Two examples

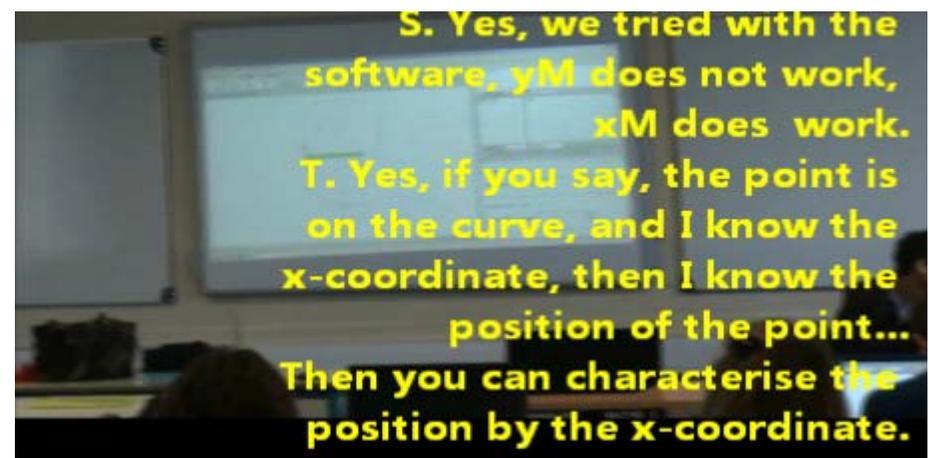
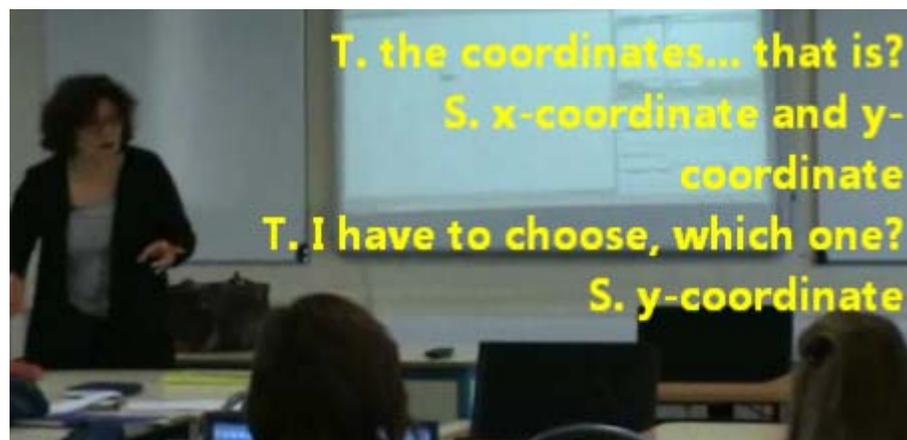
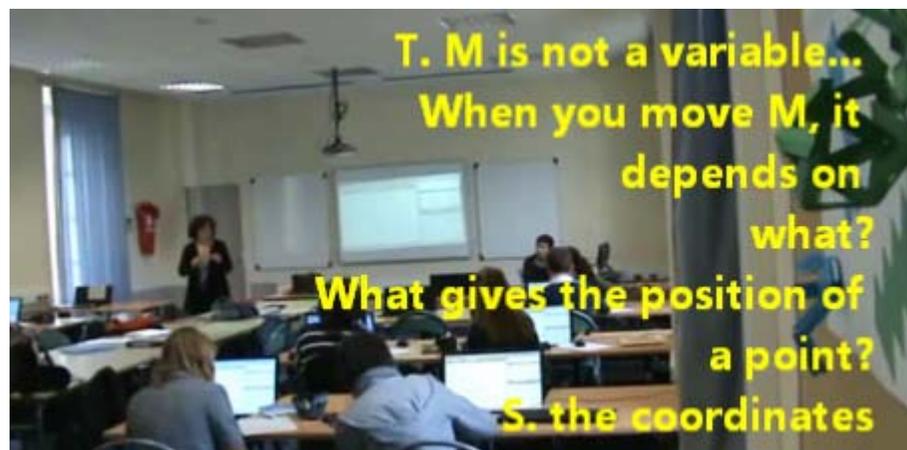
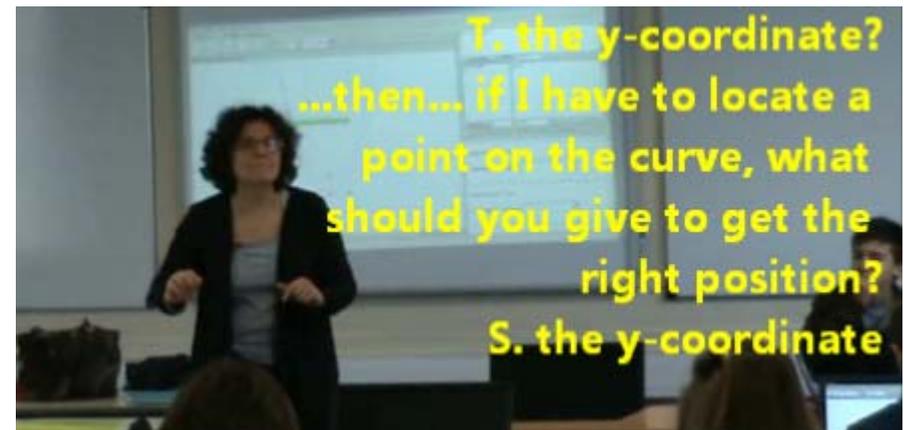
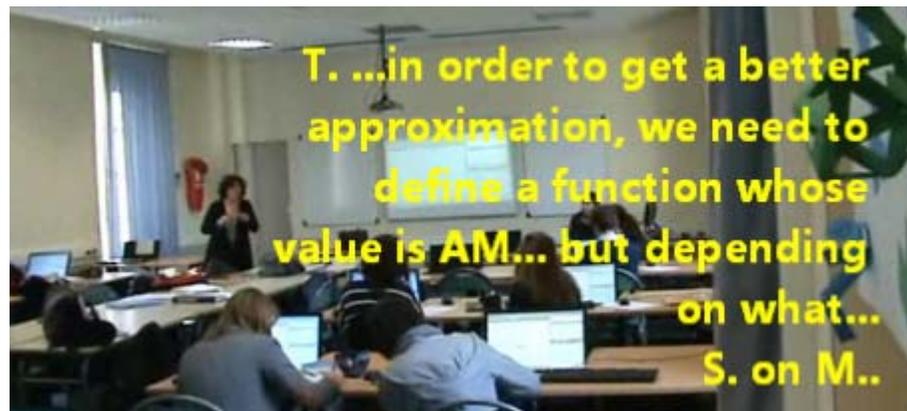
- A problem of minimum at 10th grade
- Real life experience and differentiability at 12th grade

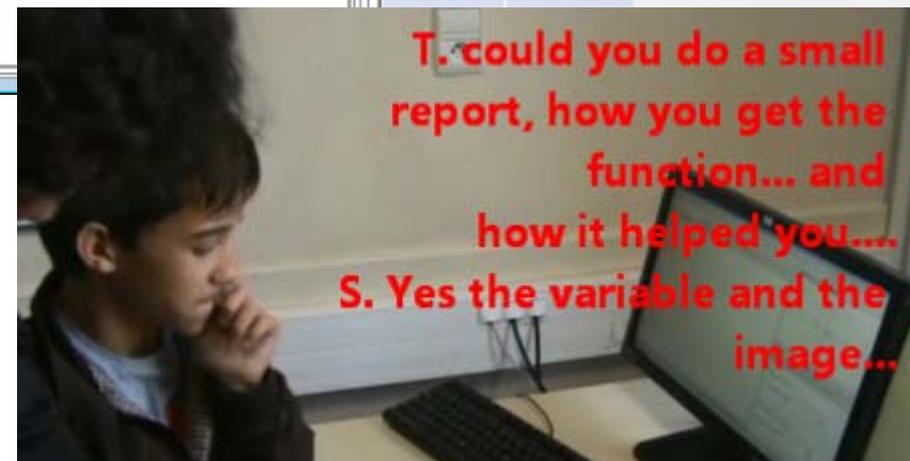
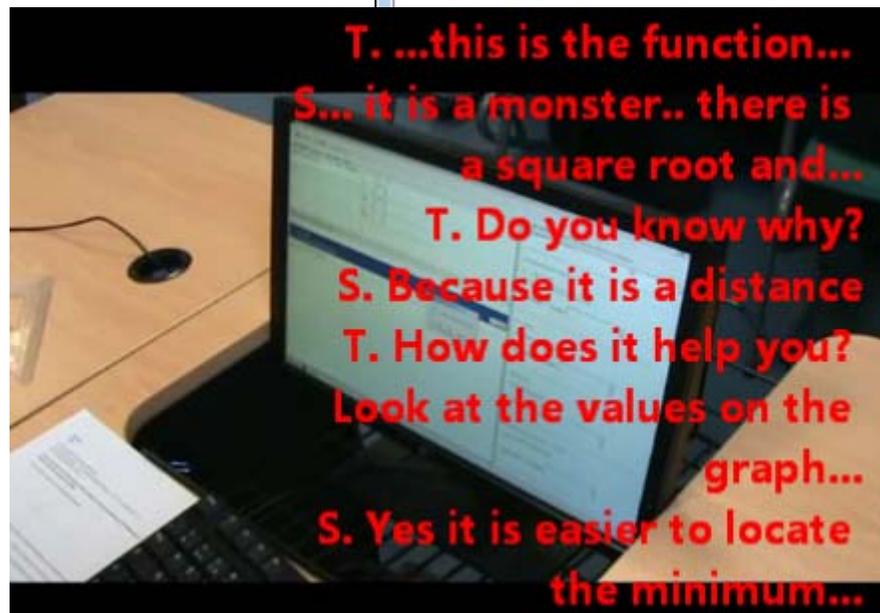
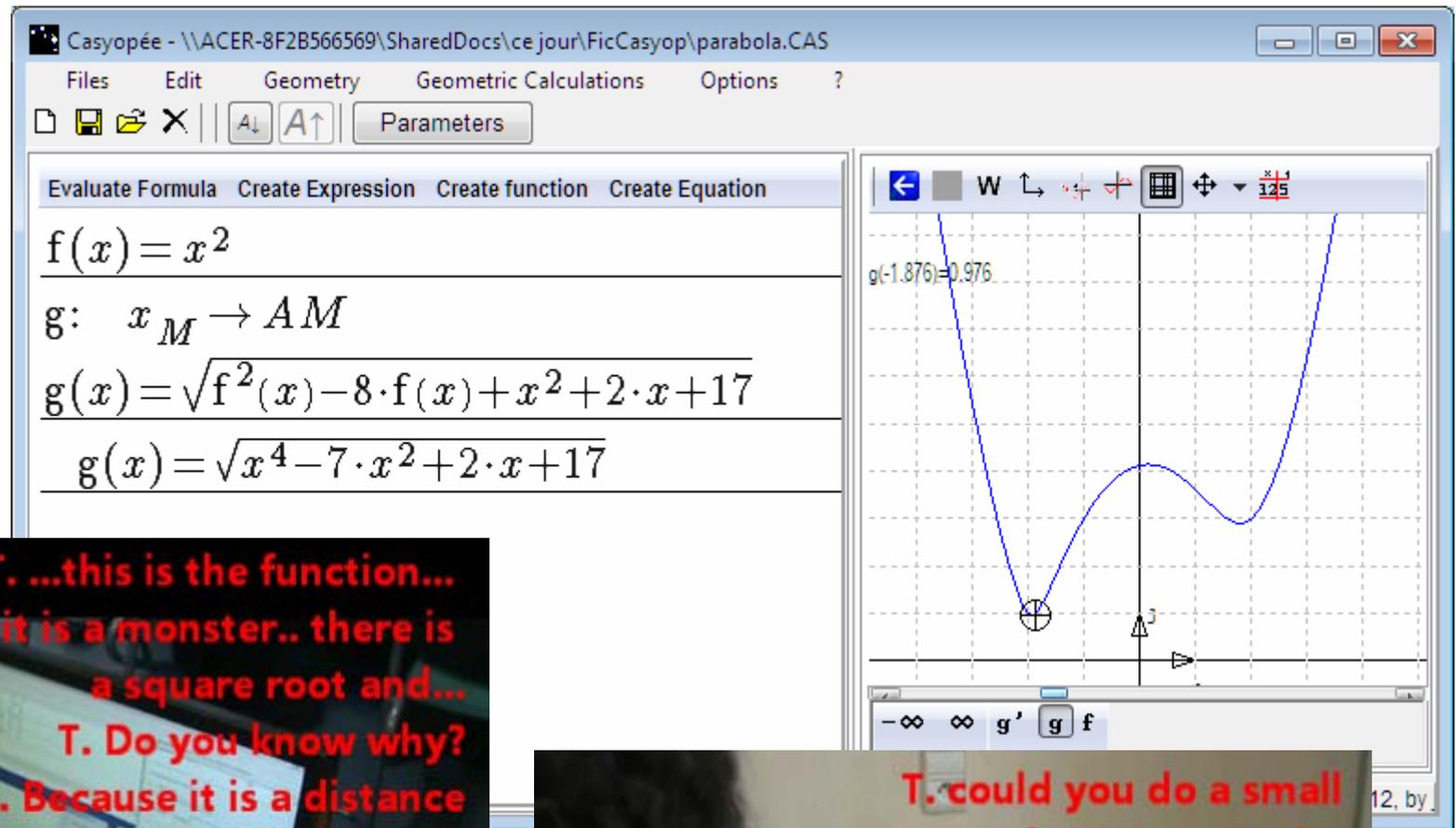
# A problem of minimum at 10th grade



- M is a point on the parabola representing  $x \rightarrow x^2$   
The goal is to find position(s) of M as close as possible to A.
- Make a dynamic geometry figure and explore.
- Use the software to propose a function modeling the problem
- Use this function to approach a solution





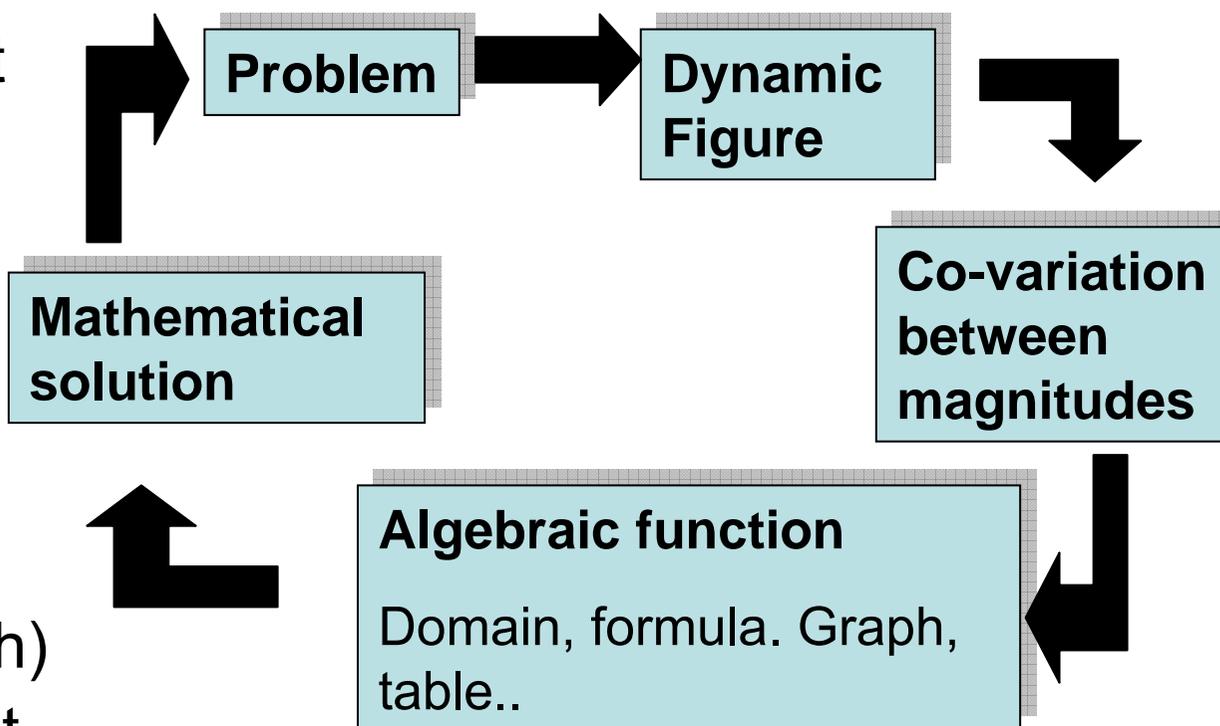


# ***Understanding co-variation***

- **Contribution of the situation**
- **Contribution of the software**

# *Contribution of the situation: a cycle of modelling*

1. From a problem to a dynamic figure
2. Identifying relevant magnitudes
3. Understanding co-variation as a functional relationship
4. Using an algebraic representation (reading on a graph)
5. Connecting a result to the problem



# Contribution of the software

- Focus on independent and dependent variables (feedback)

The screenshot displays the Casyopée software interface. The main window shows a menu bar (Files, Edit, Algebra, Graphs, Options) and a toolbar with icons for file operations and algebraic actions. A table of calculations is visible, with the following entries:

Equation	Value
$c0 = AM$	2.903
$c1 = x_M$	1.769
$c2 = y_M$ (with 'Dependence' label)	3.129

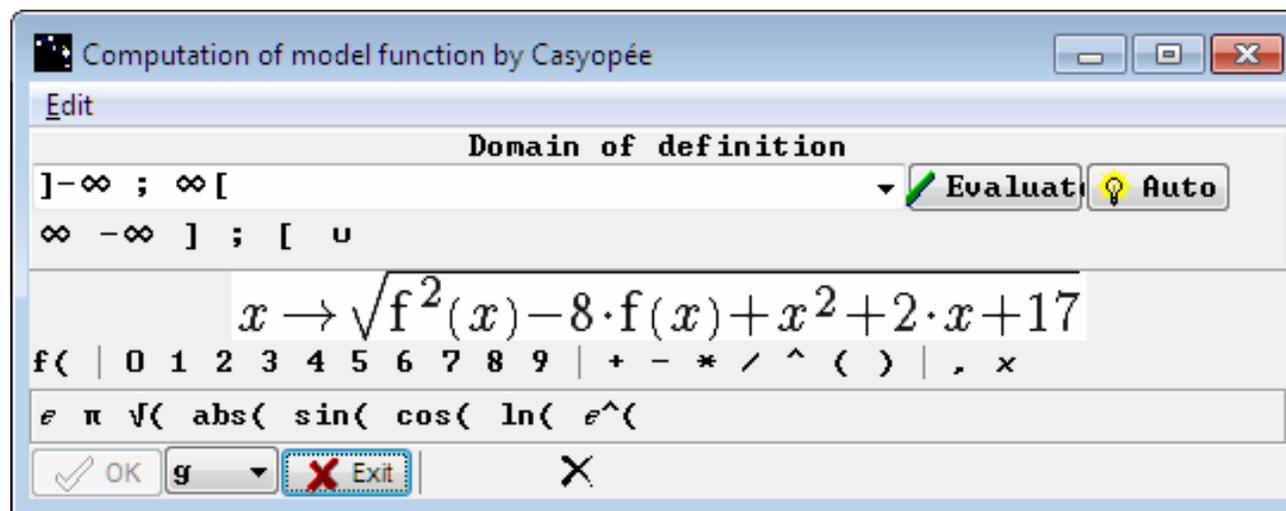
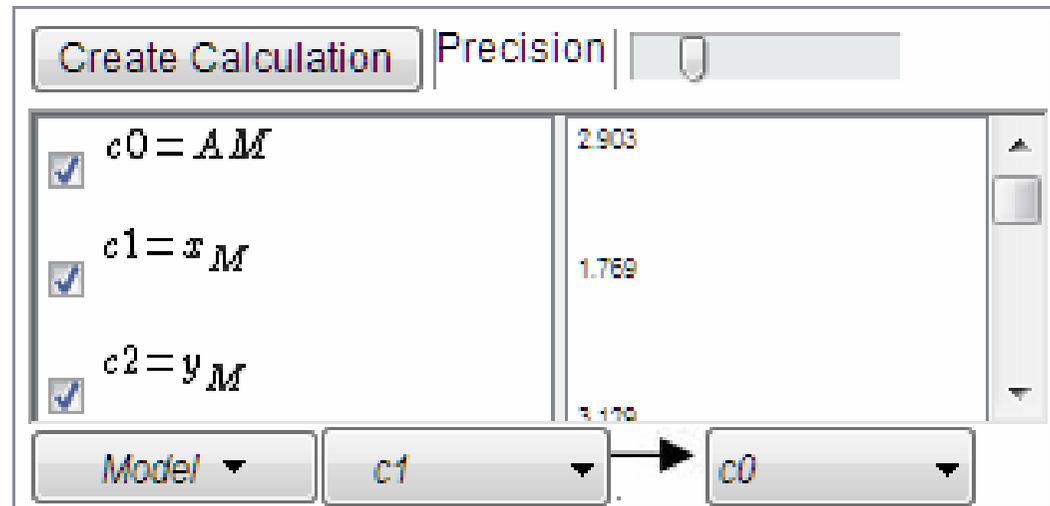
An error dialog box titled 'Casyopée' is overlaid on the main window, containing the following text:

The calculation depends on M  
Casyopée cannot compute a model function with the independent variable c2

An 'OK' button is located at the bottom right of the dialog box. The status bar at the bottom of the main window shows 'Coordinates: 3.69 5.75'.

# Contribution of the software

- Focus on the algebraic function
  - CAS feature as an help to compute functions



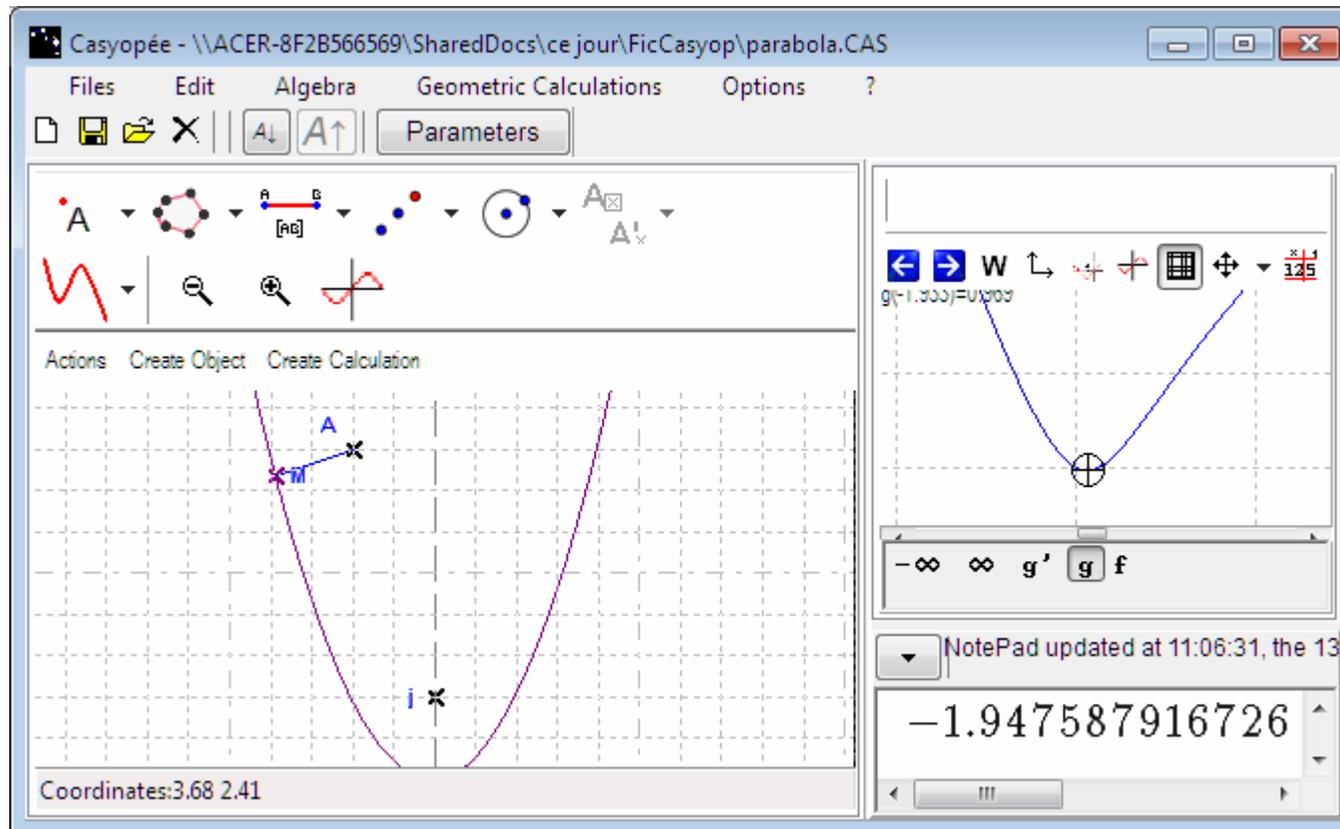
# Contribution of the software

- Focus on the algebraic function
  - A full algebraic environment Without a command language

The screenshot shows the Casyopée software interface. The window title is "Casyopée - \\ACER-8F2B566569\SharedDocs\ce jour\FicCasyop\parabola.CAS". The menu bar includes "Files", "Edit", "Geometry", "Geometric Calculations", and "Options". The toolbar contains icons for file operations and a "Parameters" button. The main workspace is divided into two panes. The left pane contains algebraic expressions: "Create Expression", "Create Equation", "Evaluate Formula", and "Options" buttons; the function  $f(x) = x^2$ ; the geometric mapping  $g: x_M \rightarrow AM$ ; the function  $g(x) = \sqrt{f^2(x) - 8 \cdot f(x) + x^2 + 2 \cdot x + 17}$ ; the simplified function  $g(x) = \sqrt{x^4 - 7 \cdot x^2 + 2 \cdot x + 17}$ ; the derivative  $g'(x) = \frac{4 \cdot x^3 - 14 \cdot x + 2}{2 \cdot \sqrt{x^4 - 7 \cdot x^2 + 2 \cdot x + 17}}$ ; and the equation  $g'(x) = 0$ . The right pane shows a graph of the function  $g(x)$  on a coordinate plane. The graph is a blue curve with a local minimum and a local maximum. A point on the curve is marked with a circle containing a plus sign. Below the graph, there are buttons for  $-\infty$ ,  $\infty$ ,  $g'$ ,  $g$ , and  $f$ . A status bar at the bottom indicates "NotePad updated at 11:06:31, the 13" and a numerical value  $-1.947587916726$ .

# Contribution of the software

- Focus on the algebraic function
  - in connection with the problem



## ***2<sup>nd</sup> example: A challenge***

- Considering “Irregular” functions
    - transition to university level
  - Connecting
    - sensual experience of movements
    - with analytic properties of model functions
- ➡ The amusement park ride:  
functional modeling and differentiability

# ***The amusement park ride: functional modeling and differentiability***

- A wheel rotates with uniform motion around its horizontal axis. A rope is attached at a point on the circumference and passes through a guide. A car is hanging at the other end.
- Motion chosen in order that a person placed in the car feel differently the transition at high and low point.



# ***The amusement park ride: objectives***

It is expected that students will

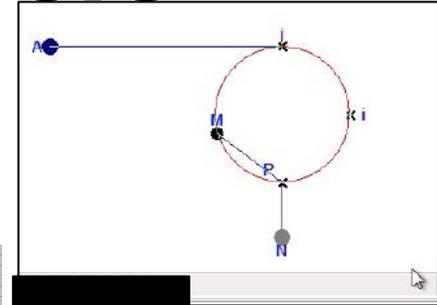
- identify the difference
- associate this with different properties of the function (non-differentiability and differentiability)
- after modelling the movement.

# The modelling cycle



Problem

Dynamic Figure

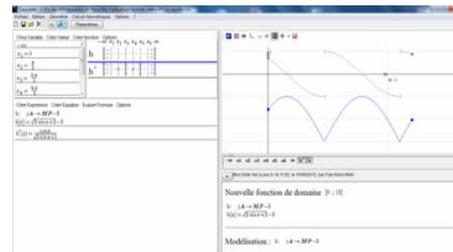
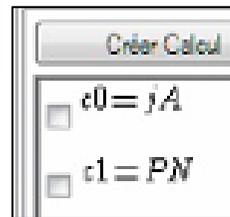
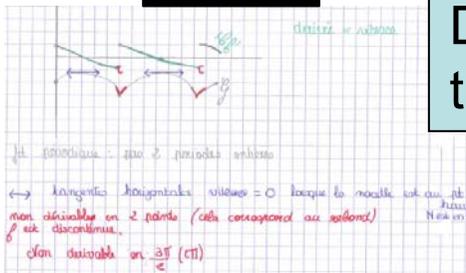


Co-variation  
between  
magnitudes

Mathematical  
solution

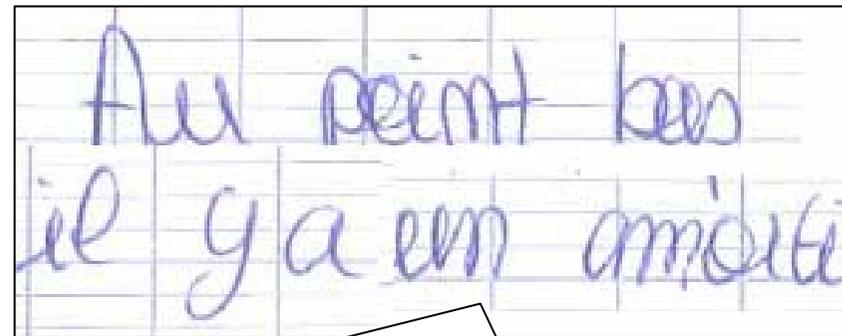
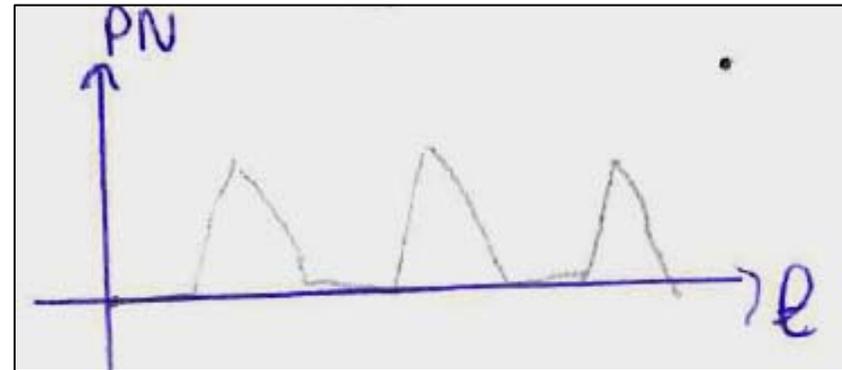
Algebraic function

Domain, formula. Graph,  
table..



# Classroom Observation

- Physical situation and spontaneous model
  - Students stick to piecewise uniform movements
  - Students are more or less aware of differences between high and low points

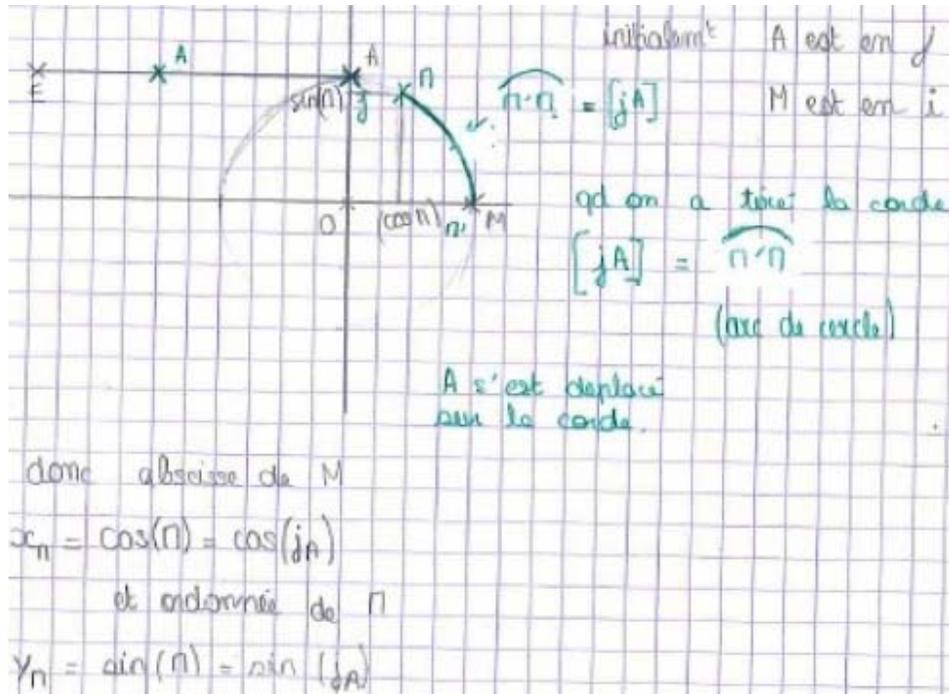


Au point bas  
il y a un amorti

At the lower point, there is a drop shot

# Building a geometric model

- Students need
  - Knowing about the artefact (dynamic geometry)
  - Associated Mathematical knowledge



3°) Le cercle a pour rayon 1, donc on travaille sur le cercle trigonométrique. Or, sur le cercle trigonométrique, la mesure d'un angle  $\widehat{AOB}$  en radians est égale à la mesure de  $\widehat{AB}$ .

Ici,  $jA = \text{mesure } \widehat{Mi} = \text{mesure } \widehat{iOM}$

donc  $(\cos(jA); \sin(jA)) \Leftrightarrow (\cos(\widehat{iOM}); \sin(\widehat{iOM}))$

$\Leftrightarrow$  Coordonnées de M sur le cercle trigonométrique  $(e; \vec{i}, \vec{j})$

# Students more or less aware of the choice of dependant and independent variables

On choisit la distance AJ en variable,  
AJ en fonction des coordonnées de N

$y = PN$  → position de la nacelle  
 $x = AJ$  → longueur de la corde tirée

We choose distance AJ as the (independent) variable

AJ is a function of the coordinates of N

$y = PN$  → the car's position

$x = AJ$  → the length of the rope drawn

# Students ignore Casyopée's warning

The image shows a screenshot of the Casyopée software interface. The main window, titled "Dérivée", displays the function  $g(x) = 2 - \sqrt{2 \cdot \sin x + 2}$  and its derivative  $g'(x) = -\frac{\cos x}{\sqrt{2 \cdot \sin x + 2}}$ . The domain of definition is set to  $[0; 12]$ . A warning dialog box is open, displaying the message: "rac(2\*sin(x)+2) : MAXIMA does not recognize a positive expression under the radical on ]x1 ; x2[. Hint: change the interval of the variable or the parameter. Ignore only if sure of the sign".

The Casyopée window shows the function  $g(x) = 2 - \sqrt{2 \cdot \sin x + 2}$  and its derivative  $g'(x) = -\frac{\cos x}{\sqrt{2 \cdot \sin x + 2}}$ . The domain of definition is set to  $x_1 = 0$  and  $x_2 = 12$ . The graph shows the function  $g(x)$  (blue curve) and its derivative  $g'(x)$  (pink curve) plotted on a coordinate system. The x-axis is labeled with  $x_1$  and  $x_2$ , and the y-axis is labeled with  $g'$  and  $g$ .

# A mediation by the teacher

The image displays three overlapping windows of the Casyopée software, illustrating a mathematical mediation process. The main window shows the definition of a function  $g$  and its derivative  $g'$ .

**Function Definition:**  
 $g: jA \rightarrow PN$   
 $g(x) = 2 - \sqrt{2 \cdot \sin x + 2}$   
 $g'(x) = -\frac{\cos x}{\sqrt{2 \cdot \sin x + 2}}$

**Graphical Representation:**  
The graph shows the function  $g$  (blue curve) and its derivative  $g'$  (red curve) over the interval  $[0, 12]$ . The derivative  $g'$  is tangent to the function  $g$  at various points. A specific point is highlighted with the coordinates  $(0.083, -0.707)$ .

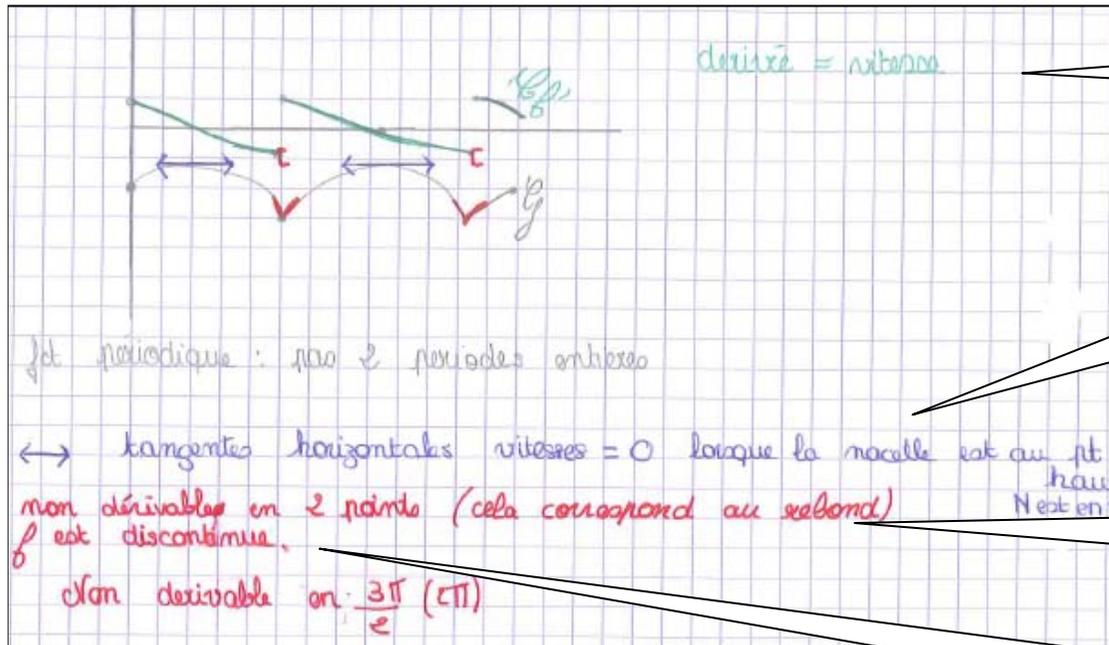
**Intermediate Window:**  
This window shows the derivative  $g'$  and a new function  $h$  defined as  $h(x) = g'(x)$ . The domain is set to  $[0, 12]$ .

**Parameter Settings:**  
The domain is defined as  $[0; 3 \cdot \pi/2 [ [0; 3 \cdot \pi/2; 7 \cdot \pi/2 [ [0; 7 \cdot \pi/2; 12]$ . The x-axis is marked with  $x_1, x_2, x_3, x_4$  and  $\infty, -\infty$ .

**Bottom Panel:**  
The bottom panel shows the x-axis with markers for  $x_1, x_2, x_3, x_4$  and  $\infty, -\infty$ . The function  $g'$  is plotted, and the derivative  $h$  is also shown. The x-axis is labeled with  $h, g', g$ .

**Footer:**  
Bloc Note mis à jour à 10:16:35, le 14/06/2012, par

# Math Solution → Problem

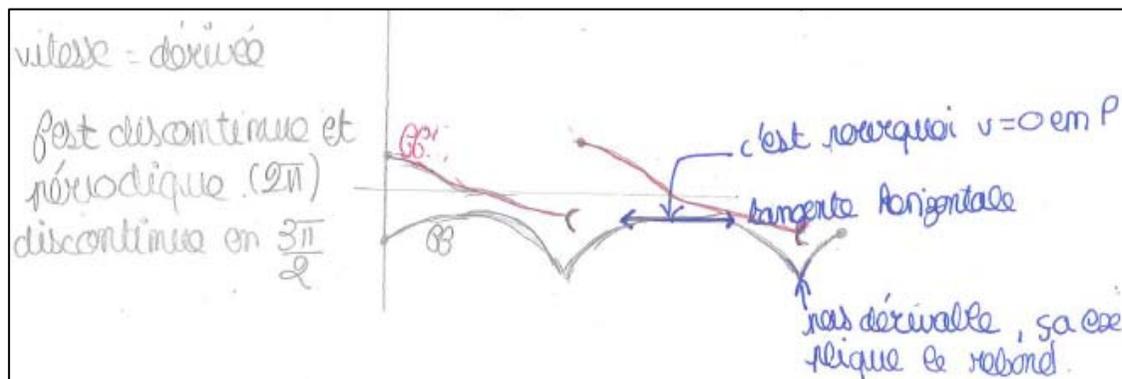


Derivative = speed

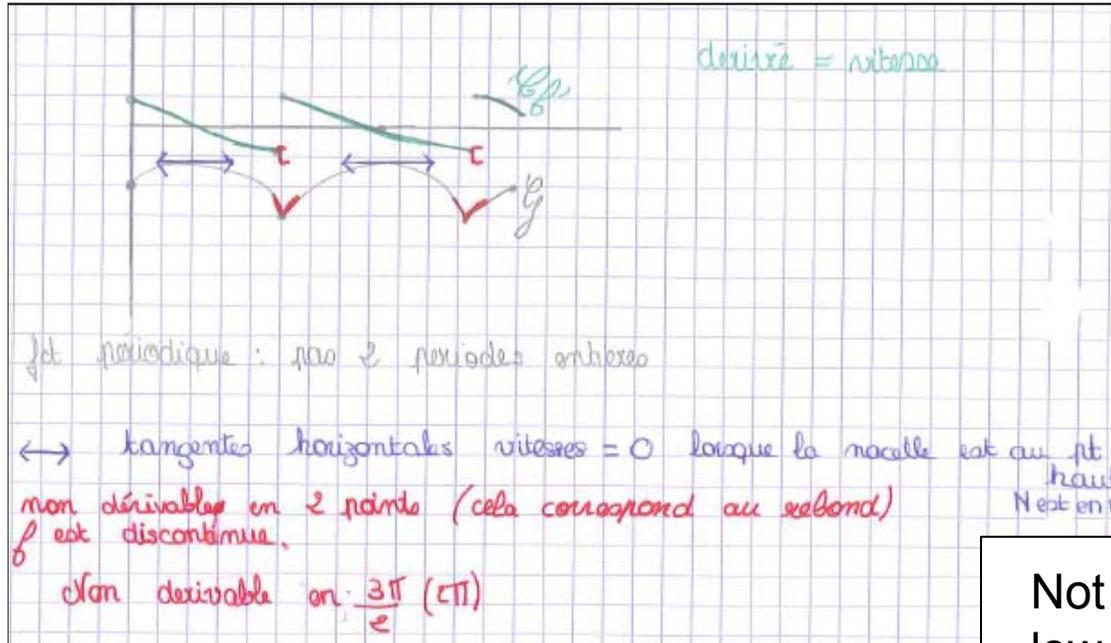
Horizontal tangent  
Speed=0 High point

Not differentiable at  
low point: rebound

The function is not  
continuous



# Math Solution → Problem

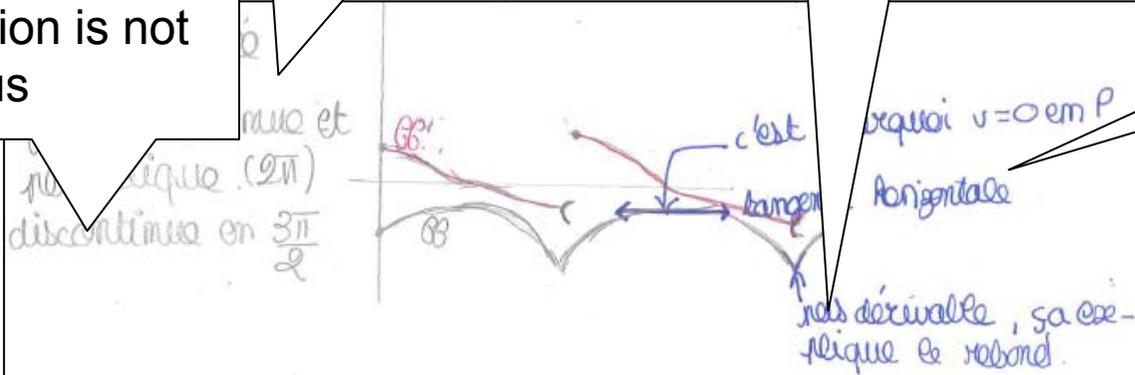


Not differentiable at low point: rebound

Derivative = speed

The function is not continuous

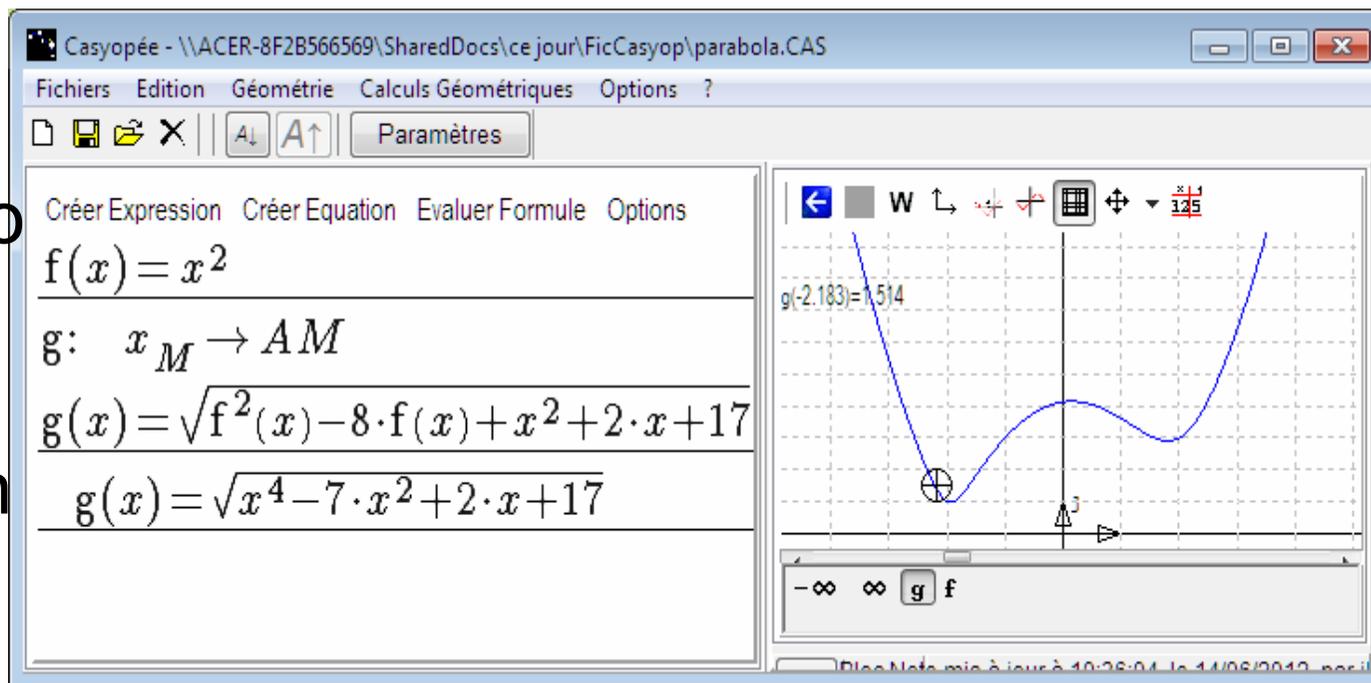
Horizontal tangent  
Speed=0 High point



# Kieran's objections

Strong presumption that symbolic forms are to be interpreted graphically, rather than dealt with directly.

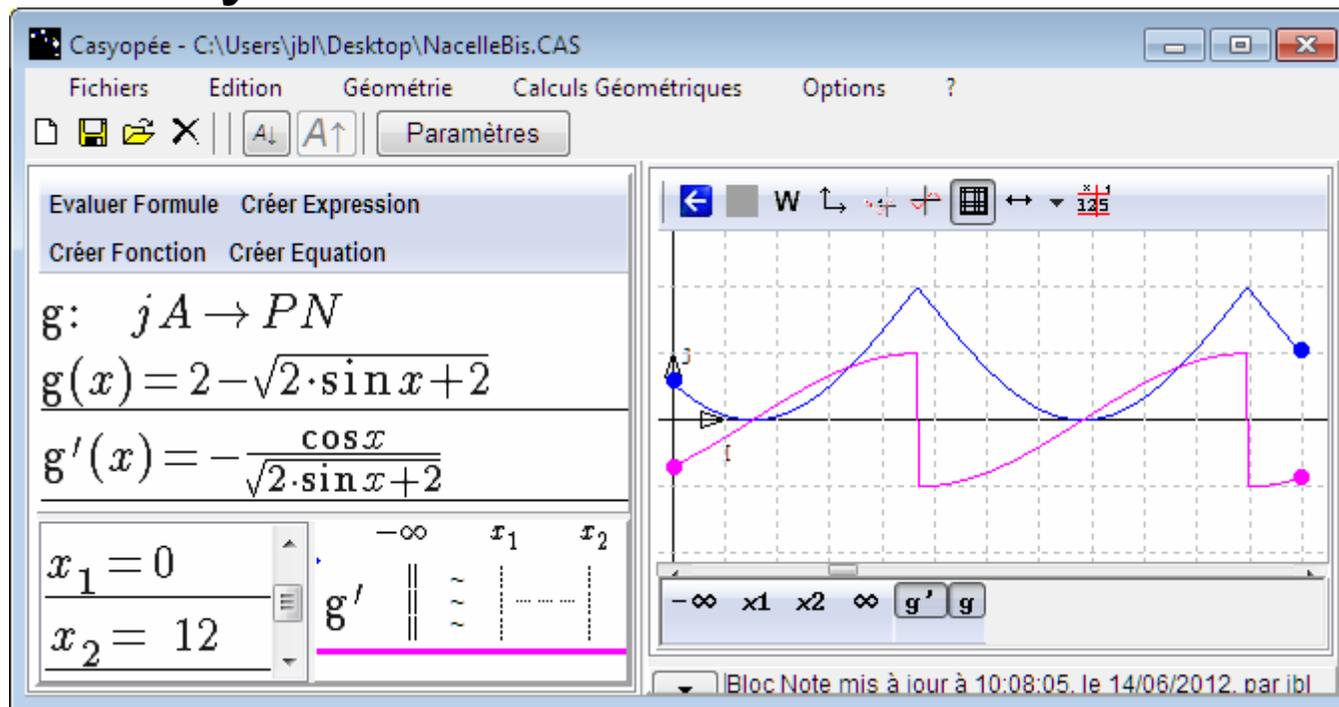
Technology being used to insist on graphical interpretation



# Kieran's objections

Strong presumption that symbolic forms are to be interpreted graphically, rather than dealt with directly.

Technology being used to insist on graphical interpretation.



# Thank you !

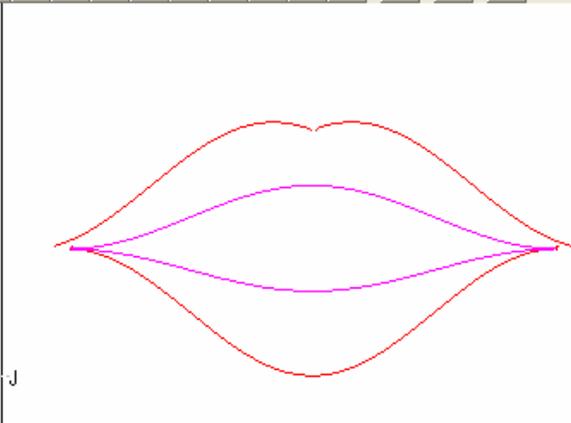
Casyopée - C:\Documents and Settings\JBL\Bureau\FicCasyop\bise.CAS

Fichiers Edition Créer Calculer Justifier Exploration Options ?

Fonc Expr Equa  $\frac{a}{b} = \dots$  bloc notes  $\text{Aa}$   $\text{Aa}$  Géométrie Dynamique

$-\infty$   
  $x_1 = \frac{19}{10} - \frac{\pi}{2}$   
  $x_2 = 2 - \frac{\pi}{2}$   
  $x_3 = 2$   
  $x_4 = \frac{\pi}{2} + 2$   
  $x_5 = \frac{\pi}{2} + \frac{21}{10}$   
  $\infty$

<input checked="" type="checkbox"/> g	~ ~ ~ ~ ~
<input type="checkbox"/> h	~ ~ ~ ~ ~
<input type="checkbox"/> k	~ ~ ~ ~ ~
<input checked="" type="checkbox"/> l	~ ~ ~ ~ ~
<input checked="" type="checkbox"/> m	~ ~ ~ ~ ~
<input checked="" type="checkbox"/> p	~ ~ ~ ~ ~



<http://casyopee.eu>

$f(x) = \cos\left(2x + \frac{5}{10} - 4\right) + 4$   
 $g(x) = f(4 - x)$   
 $h(x) = -f(x) + 6$   
 $k(x) = -g(x) + 6$   
 $l(x) = \frac{1 + \cos(2x - 4)}{2} + 3$   
 $m(x) = -\frac{2}{3} \cdot l(x) + 5$   
 $p(x) = 3 \cdot m(x) - 6$

$p(x) = 2 \cdot m(x)$