Quasiorder lattices of varieties

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Definition

The set of **compatible quasiorders** of an algebra **A** is

$$\mathrm{Quo}(\mathbf{A}) = \{\, \alpha \leq \mathbf{A}^2 \mid \alpha \text{ is reflexive and transitive} \,\}.$$

1. A quasiorder $\alpha \subseteq A^2$ is compatible with **A** if
$$(x, y) \in \alpha \implies (p(x), p(y)) \in \alpha$$
for all unary polinomials p of **A**.
2. $\mathrm{Quo}(\mathbf{A})$ forms an (involution) lattice with $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee \beta = \overline{\alpha \cup \beta}$, where $\overline{\alpha \cup \beta}$ is the transitive closure of $\alpha \cup \beta$.
3. The set $\mathrm{Con}(\mathbf{A})$ of congruences forms a sublattice of $\mathrm{Quo}(\mathbf{A})$.

Goal

Systematic study of the connection between congruence identities, quasiorder identities and Maltsev conditions satisfied by varieties.

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Why study compatible quasiorders?

1. More general than congruences.
2. Better behaved than tolerances.
3. Some connection with the constraint satisfaction problem:

For a subdirect power $\mathbf{R} \leq_{\mathrm{sd}} \mathbf{A}^n$ and a closed path

$$p := k_1 \to k_2 \to \cdots \to k_m \to k_1 \quad \text{with} \quad k_i \in \{1, \ldots, n\}$$

define

$$\alpha_p = \bigcup_{i=1}^{\infty} (\eta_{k_1} \circ \eta_{k_2} \circ \cdots \circ \eta_{k_m})^i \quad \text{where} \quad \eta_k = \ker \pi_k.$$

We have $\alpha_p \in \mathrm{Quo}(\mathbf{R})$ and $\alpha_p \vee \eta_{k_1}$ can be computed from the following two-projections:

$$\pi_{k_1 k_2}(R),\ \pi_{k_2 k_3}(R), \ldots, \pi_{k_m k_1}(R).$$

"Prague strategy" iff $\mathrm{range}(p) \subseteq \mathrm{range}(q) \implies \alpha_p \leq \alpha_q$.

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Is this study interesting?

Main results:

1. A locally finite variety $\mathcal{V}$ is congruence distributive ($\mathrm{Con}(\mathbf{A})$ is distributive for all $\mathbf{A} \in \mathcal{V}$) if and only if it is quasiorder distributive ($\mathrm{Quo}(\mathbf{A})$ is distributive for all $\mathbf{A} \in \mathcal{V}$).
2. A locally finite variety is congruence modular if and only if it is quasiorder modular.
3. The variety of semilattices is not quasiorder meet semi-distributive (but it is congruence meet semi-distributive).
4. $\mathrm{Quo}(\mathbf{A})$ is not in the lattice quasivariety generated by the congruence lattices $\mathrm{Con}(\mathbf{B})$ for $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$.
5. For a finite algebra $\mathbf{A}$ in a congruence meet semi-distributive variety $\mathrm{Quo}(\mathbf{A})$ has no sublattice isomorphic to $\mathbf{M}_3$.
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Congruence distributivity

Theorem (B. Jónsson, 1967)

A variety is congruence distributive iff it has Jónsson terms

$$x \approx p_1(x,x,y) \text{ and } p_n(x,y,y) \approx y,$$
$$p_i(x,y,y) \approx p_{i+1}(x,y,y) \text{ for odd } i,$$
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Theorem (G. Czédli and A. Lenkehegyi, 1983; I. Chajda, 1991)

There is a Maltsev condition charaterizing quasiorder distributivity.

Corollary (G. Czédli and A. Lenkehegyi, 1983)

If a variety $\mathcal{V}$ has a majority term, then it is quasiorder distributive.

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Directed Jónsson terms

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The ternary terms $p_1, \ldots, p_n$ are **directed Jónsson terms** if

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A variety is congruence distributive if and only if it has directed Jónsson terms.

Lemma (A. Kazda, M. Kozik, R. McKenzie and M. Moore, 2014)

If $\alpha \lhd_{\mathrm{WJ}} \beta$ (weak Jónsson absorbs) for $\alpha, \beta \in \mathrm{Quo}(\mathbf{A})$ then $\alpha = \beta$.

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Theorem (L. Barto, 2012)

Finitely related algebras in congruence distributive varieties have near unanimity terms.

$$t(y, x, \ldots, x) \approx t(x, y, x \ldots, x) \approx \cdots \approx t(x, \ldots, x, y) \approx x.$$

Theorem

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Proof.

Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$ be the two-generated free algebra, and put

$$R = \mathrm{Sg}\{(x, x, x), (x, y, y), (y, x, y)\} \leq \mathbf{F}^3.$$

The algebra $(F; \mathrm{Pol}(R))$ is finitely related and has Jónsson terms, so R has a near-unanimity polymorphism t. The terms generating the tuples $t((y, x, y), \ldots, (y, x, y), (x, y, y), (x, x, x), \ldots, (x, x, x))$ are directed Jónsson terms. □

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Theorem

If a finite algebra has directed Jónsson terms, then it is quasiorder distributive.

Proof.

1. We show $(\alpha \vee \beta) \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ for $\alpha, \beta, \gamma \in \mathrm{Quo}(\mathbf{A})$
2. Put $\gamma^* = \gamma \cap \gamma^{-1} \in \mathrm{Con}(\mathbf{A})$
3. Choose $(a, b) \in (\alpha \vee \beta) \wedge \gamma - (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ such that the interval $[a/\gamma^*, b/\gamma^*]$ is minimal in the poset $(A/\gamma^*; \gamma/\gamma^*)$
4. We have a chain of $\alpha \cup \beta$ links connecteing a and b
5. Use the directed Jónsson terms to move this chain inside the interval $[a, b] = \{ x \mid a \; \gamma \; x \; \gamma \; b \}$.
6. The links inside a/γ^* are in $(\alpha \wedge \gamma) \cup (\beta \wedge \gamma)$.
7. The first link leaving a/γ^* is also in $(\alpha \wedge \gamma) \cup (\beta \wedge \gamma)$.
8. By minimality the rest is also in $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$. □

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Does the above equivalence hold for all varieties? Does quasiorder distributivity imply directed Jónsson terms syntactically?

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For a finite algebra with directed Jónsson terms and α, β compatible reflexive relations we have $\overline{\alpha} \cap \overline{\beta} = \overline{\alpha \cap \overline{\beta}}$.

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Directed Gumm terms

Definition

The ternary terms $p_1, \dots, p_n, q$ are **directed Gumm terms** if

$$x \approx p_1(x, x, y),$$
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Theorem (A. Kazda, M. Kozik, R. McKenzie and M. Moore, 2014)

A variety is congruence modular if and only if it has directed Gumm terms.

- Has been known for locally finite varieties (M. Kozik)
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If a finite algebra has directed Gumm terms then the lattice of its compatible quasiorders is modular.

- To show $\alpha \leq \gamma \implies (\alpha \vee \beta) \wedge \gamma \leq \alpha \vee (\beta \wedge \gamma)$ we take again a counterexample pair (a, b) with minial distance in γ/γ^*.
- Significantly harder than the distributive case.

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In n-permutable varieties compatible quasiorders are congruences.

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The dual condition is **congruence join semi-distributivity**.

Typical meet semi-distributive variety is the variety of semilattices (or varieties with totally symmetric operations of all arities).

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The variety of semilattices is not quasiorder meet semi-distributive.

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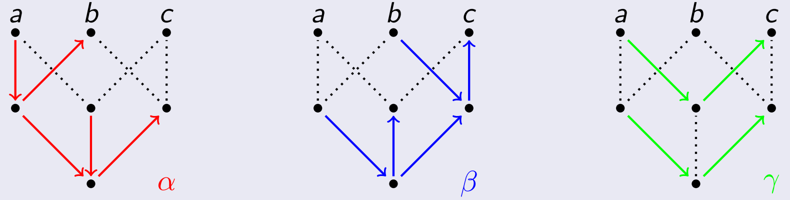

Theorem (D. Hobby and R. McKenzie, TCT Theorem 9.10)

For any locally finite variety $\mathcal{V}$ the following are equivalent:

1. $\mathrm{typ}\{\mathcal{V}\} \cap \{\mathbf{1}, \mathbf{2}\} = \emptyset$.
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- The previous example shows that $\mathbf{D}_1$ is a sublattice of the quasiorder lattice of the free semilattice with three generators.
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The very beginning of tame congruence theory (excluding the classification of minimal algebras) goes through.

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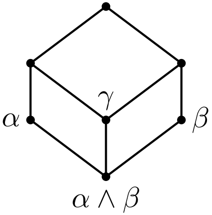

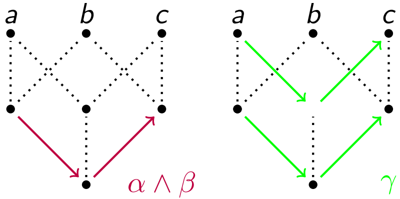

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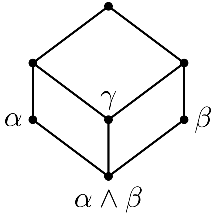

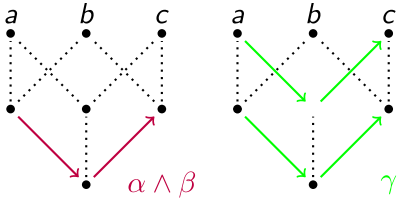

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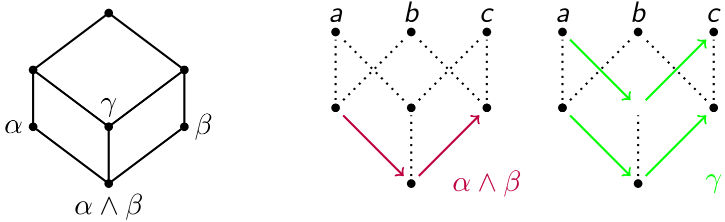

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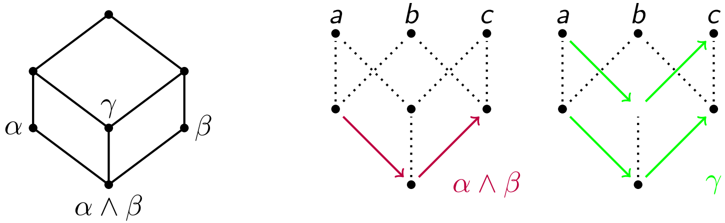

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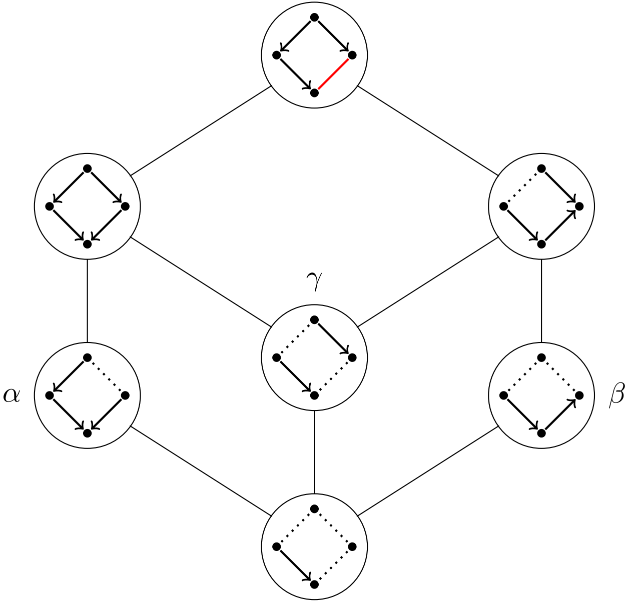
- We can take the image of **S** under the idempotent polynomial $p(x) = a \wedge x$.
- We have $p(\gamma) \not\subseteq \alpha \wedge \beta$ so p embeds the $\mathbf{D}_1$ sublattice into the quasiorder lattice of $p(\mathbf{A})$.



γ
α
β

Theorem

For a finite algebra $\mathbf{A}$ *in a congruence meet semi-distributive variety* $\mathrm{Quo}(\mathbf{A})$ *does not have a sublattice isomorphic to* $\mathbf{M}_3$.

Proof.

1. Choose a minimal sublattice of $\mathrm{Quo}(\mathbf{A})$ isomorphic to $\mathbf{M}_3$.
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- We (hope to) have a construction of an infinite semilattice whose lattice of compatible quasiorders has an $\mathbf{M}_3$ sublattice.
- Working on congruence join semi-distributivity and omitting the $\mathbf{M}_3$ and $\mathbf{D}_2$ sublattices.
- Trying to find a good notion of the commutator for quasiorders (if there is such a thing).
- Interesting theorems in the TCT book (e.g. Theorem 5.26) about orderable tame quotients (types **4** and **5**) and (α, β)-preorders.

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Thank You!