Orthomodular Posets Can Be Organized as Conditionally Residuated Structures

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Abstract

It is proved that orthomodular posets are in a natural one-to-one correspondence with certain residuated structures.

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Orthomodular posets are well-known structures used in the foundations of quantum mechanics (cf. e.g. [4], [5], [9], [10] and [11]). They can be considered as effect algebras (see e.g. [6]). Residuated lattices were treated in [7]. In [3] the concept of a conditionally residuated structure was introduced. Since every orthomodular poset is in fact an effect algebra, it follows that also every orthomodular poset can be considered as a conditionally residuated structure. The question is which additional conditions have to be satisfied in order to get a one-to-one correspondence. Contrary to the case of effect algebras, orthomodular posets satisfy also the orthomodular law and a certain condition concerning the orthogonality of their elements.
We start with the definition of an orthomodular poset.

Definition 1

An *orthomodular poset* (cf. [8], [2] and [12]) is an ordered quintuple $\mathcal{P} = (P, \leq, ^\perp, 0, 1)$ where $(P, \leq, 0, 1)$ is a bounded poset, $^\perp$ is a unary operation on P and the following conditions hold for all $x, y \in P$:

(i) $(x^\perp)^\perp = x$

(ii) If $x \leq y$ then $y^\perp \leq x^\perp$.

(iii) If $x \perp y$ then $x \vee y$ exists.

(iv) If $x \leq y$ then $y = x \vee (y \wedge x^\perp)$.

Here and in the following $x \perp y$ is an abbreviation for $x \leq y^\perp$.

Remark 1

If $(P, \leq)$ is a poset and $^\perp$ a unary operation on P satisfying (i) and (ii) then the so-called de Morgan laws

$$(x \vee y)^\perp = x^\perp \wedge y^\perp \text{ in case } x \perp y \text{ and}$$
$$(x \wedge y)^\perp = x^\perp \vee y^\perp \text{ in case } x^\perp \perp y^\perp$$

hold. Moreover, (iv) is equivalent to the following condition:

(v) If $x \leq y$ then $x = y \wedge (x \vee y^\perp)$.

If $x \leq y$ then $x \perp y^\perp$ and therefore $x \vee y^\perp$ is defined. Hence also $y \wedge x^\perp$ is defined. Moreover, $x \perp y \wedge x^\perp$ which shows that $x \vee (y \wedge x^\perp)$ is defined. Thus the expression in (iv) is well-defined. The same is true for condition (v).

Next we define a partial commutative groupoid with unit.

Definition 2

A *partial commutative groupoid with unit* is a partial algebra $\mathcal{A} = (A, \odot, 1)$ of type $(2, 0)$ satisfying the following conditions for all $x, y \in A$:

(i) If $x \odot y$ is defined so is $y \odot x$ and $x \odot y = y \odot x$.

(ii) $x \odot 1$ and $1 \odot x$ are defined and $x \odot 1 = 1 \odot x = x$.

Now we are ready to define a conditionally residuated structure.

Definition 3

Let $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ be an ordered sixtuple such that $(A, \leq, 0, 1)$ is a bounded poset, $(A, \odot, \to, 0, 1)$ is a partial algebra of type $(2, 2, 0, 0)$, $(A, \odot, 1)$ is a partial commutative groupoid with unit and $x \to y$ is defined if and only if $y \leq x$. We write x' instead of $x \to 0$. Moreover, assume that the following conditions are satisfied for all $x, y, z \in A$:

(i) $x \odot y$ is defined if and only if $x' \leq y$.

(ii) If $x \odot y$ and $y \to z$ are defined then $x \odot y \leq z$ if and only if $x \leq y \to z$.

(iii) If $x \to y$ is defined then so is $y' \to x'$ and $x \to y = y' \to x'$.

(iv) If $y \leq x$ and $x', y \leq z$ then $x \to y \leq z$.

Then $\mathcal{A}$ is called a *conditionally residuated structure*.

Remark 2

Condition (ii) is called *left adjointness*, see e.g. [1].

Example 1

Let $M := \{1, \ldots, 6\}$ and $P := \{C \subseteq M \mid |C| \text{ is even}\}$. If one defines for arbitrary $A, B \in P$

$$
\begin{aligned}
& A \odot M = M \odot A := A, \\
& A \odot (M \setminus A) := \emptyset, \\
& A \odot B := A \cap B \text{ if } |A| = |B| = 4 \text{ and } A \cup B = M, \\
& A \to \emptyset := M \setminus A, \\
& A \to A := M, \\
& M \to A := A \text{ and} \\
& A \to B := (M \setminus A) \cup B \text{ if } B \subseteq A, |B| = 2 \text{ and } |A| = 4
\end{aligned}
$$

then $(P, \subseteq, \odot, \to, \emptyset, M)$ is a conditionally residuated structure.

The following lemma lists some easy properties of conditionally residuated structures used later on.

Lemma 1

If $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure then the following conditions hold for all $x, y \in A$:

(i) $(x')' = x$

(ii) If $x \leq y$ then $y' \leq x'$.

(iii) If $x \odot y$ is defined then $x \odot y = 0$ if and only if $x \leq y'$.

(iv) $x \to y = 1$ if and only if $x \leq y$.

We now introduce two more properties of conditionally residuated structures.

Definition 4

A conditionally residuated structure $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ is said to satisfy the *divisibility condition* if $y \leq x$ implies that $x \odot (x \rightarrow y)$ exists and $x \odot (x \rightarrow y) = y$ and it is said to satisfy the *orthogonality condition* if $x \leq y'$, $y \leq z'$ and $z \leq x'$ together imply $z \leq x' \odot y'$.

In the following theorem we show that an orthomodular poset can be considered as a special conditionally residuated structure.

Theorem 1

If $\mathcal{P} = (P, \leq, ^{\perp}, 0, 1)$ is an orthomodular poset and one defines

$$
\begin{aligned}
x \odot y &:= x \wedge y \text{ if and only if } x^{\perp} \leq y \text{ and} \\
x \to y &:= x^{\perp} \vee y \text{ if and only if } y \leq x
\end{aligned}
$$

for all $x, y \in P$ then $\mathbf{A}(\mathcal{P}) := (P, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure satisfying both the divisibility and orthogonality condition.

Conversely, we show that certain conditionally residuated structures can be converted in an orthomodular poset.

Theorem 2

If $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure satisfying the divisibility and orthogonality condition then $\mathbf{P}(\mathcal{A}) := (A, \leq, ', 0, 1)$ is an orthomodular poset.

Finally, we show that the correspondence described in the last two theorems is one-to-one.

Theorem 3

If $\mathcal{P} = (P, \leq, ^{\perp}, 0, 1)$ is an orthomodular poset then $\mathbf{P}(\mathbf{A}(\mathcal{P})) = \mathcal{P}$.
If $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ is a conditionally residuated structure satisfying the divisibility and orthogonality condition then $\mathbf{A}(\mathbf{P}(\mathcal{A})) = \mathcal{A}$.

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