A Birkhoff's theorem for varieties defined by linear equations

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Linear equations

Given a signature Σ, a term t over Σ is linear if it contains at most one operation symbol from Σ. E.g.

$$t(x_0, x_1, x_2, x_3) = f(x_1, x_0, x_0, x_2)$$

where f is a basic symbol, or a projection.

An equation is linear if both sides are linear terms.

Example

$$f(x_{i_1}, \dots, x_{i_n}) \approx g(x_{j_1}, \dots, x_{i_m}), \quad \text{or} \quad f(x_{i_1}, \dots, x_{i_n}) \approx x_j.$$

Motivation

CSP dichotomy conjecture

Let (A, Γ) be a finite relational structure. Then $\mathrm{CSP}(A, \Gamma)$ is tractable iff $\mathrm{Pol}(A, \Gamma)$ contains an operation s satisfying

$$s(x,x,x,y,y,y) \approx s(x,y,y,x,x,y) \approx s(y,x,y,x,y,x).$$

Many of naturally apearing Mal'cev conditions are linear, for example Mal'cev term, near unanimity term, cube terms, $\mathrm{SD}(\wedge)$, Gumm term, Jónsson terms, etc.

Another Valeriote's conjecture

For every strong linear idempotent Mal'cev condition there is a poly-time algorithm that decides whether an idempotent algebra $\mathbf{A}$ satisfies this condition.

The question

Birhoff's HSP theorem

Let $\mathcal{K}$ be a class of algebras of a given signature. Then

$$\operatorname{Mod}\operatorname{Eq}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K}).$$

Linear Birkhoff theorem

Let $\mathcal{K}$ be a class of algebras of a given signature. Then

$$\operatorname{Mod}\operatorname{LinEq}(\mathcal{K}) = \ \mathbf{?}\ (\mathcal{K}).$$

Retractions

An algebra $\mathbf{B}$ is a retraction of $\mathbf{A}$ if there are maps

$$\mathbf{A} \underset{a}{\overset{b}{\rightleftarrows}} \mathbf{B}$$

such that $ba = 1_B$, and for every operation f we have

$$f_{\mathbf{B}}(b_0, \ldots, b_{n-1}) = bf_{\mathbf{A}}(a(b_0), \ldots, a(b_{n-1})).$$

Observation

If $\mathbf{B}$ is a retraction of $\mathbf{A}$ then it satifies all the linear equations valid in $\mathbf{A}$.

The class of all retractions of algebras from $\mathcal{K}$ will be denoted $\mathbf{R}(\mathcal{K})$.

Note that $\mathbf{HS} \subseteq \mathbf{R}$.

The question

Birhoff's HSP theorem

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Linear Birkhoff theorem

Let $\mathcal{K}$ be a class of algebras of a given signature. Then

$$\operatorname{Mod}\operatorname{LinEq}(\mathcal{K}) = \ ? \ \mathbf{RP}(\mathcal{K}).$$

Proof of ‘Mod LinEq $\subseteq$ **RP**’

Let $\mathbf{B} \in \operatorname{Mod}\operatorname{LinEq}(\mathcal{K})$, $\mathcal{V} = \mathbf{HSP}(\mathcal{K})$, and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(B)$.
Observe that $\operatorname{LinEq}(\mathbf{F}) \subseteq \operatorname{LinEq}(\mathcal{K})$.

Take $f\colon B \to F$ to be 1_B, and define $b\colon F \to B$ as

$$b(f) = \begin{cases} t_{\mathbf{B}}(b_0, \ldots, b_{n-1}) & \text{if } f = t_{\mathbf{F}}(b_0, \ldots, b_{n-1}) \text{ for some linear term } t \\ & \text{and } b_0, \ldots, b_{n-1} \in B, \\ \text{whatever in } B & \text{otherwise.} \end{cases}$$

Finally, $\mathbf{B}$ is (f, b)-retraction of $\mathbf{F}$. So,

$$\mathbf{B} \in \mathbf{R}(\mathbf{F}) \subseteq \mathbf{RHSP}(\mathcal{K}) = \mathbf{RP}(\mathcal{K})$$

□

Applications

Proposition

The following Mal'cev conditions are not equivalent to a linear one.

- *congruence regularity,*
- *congruence singularity,*
- *congruence permutability* $\wedge$ *congruence distributivity.*

Applications (cont.)

Theorem

An algebra $\mathbf{A}$ *doesn't ... if and only if there is an algebra* $\mathbf{X}$ *such that ... and* $\mathbf{X} \in \mathbf{RP}(\operatorname{Clo}\mathbf{A})$.

- *... have a Mal'cev term ...*
  $\dots X = \{0,1,2\}$, $\{01|2, 0|12\} \subseteq \operatorname{Con}\mathbf{X}$ ...
- *... generate congruence modular variety ...*
  $\dots X = \{0,1,2,3\}$, $\{12|34, 13|24, 1|2|34\} \subseteq \operatorname{Con}\mathbf{X}$ ...
- *... have a k-cube term ...*
  $\dots X = \{0,1\}^k \setminus \{(1,\dots,1)\}$, $\operatorname{Ker} \mathit{proj}_i \in \operatorname{Con}\mathbf{X}$ *for all i* ...
- *... generate a variety satisfying a non-trivial congruence identity ...*
  $\dots X = \{0,1,2,3\}$, $\{12|34, 13|24, 1|234\} \subseteq \operatorname{Con}\mathbf{X}$ ...

Thank you for your attention!