Extending the Blok-Esakia theorem

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Intuitionistic logic: obtained from classical logic by dropping excluded middle $p \vee \neg p$

more formally
Int: the set of formulas containing "some" axioms and closed under modus ponens.

Intermediate logics: sets containing **Int** and closed under modus ponens and substitutions minus the trivial logic
("everything" between **Cl** and **Int**)

Very Very Old Problem

Understand $\rightarrow$ in intermediate logics

S4 modal logic

S4: (more or less) classical logic augmented by $\Box$ s.t.

$$1 \leftrightarrow \Box 1,\ \Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q,\ \Box\Box p \leftrightarrow \Box p \to p \ \in \ \mathbf{S4}$$

Normal extensions of **S4**: extensions closed under modus ponens necessitation $(\alpha/\Box\alpha)$ and substitutions

Solution to VVOP (conj. by Gödel, proved by McKinsey Tarski)

$$\alpha \in \mathbf{Int} \quad \iff \quad \mathrm{Tr}(\alpha) \in \mathbf{S4}$$

$\mathrm{Tr}(\alpha)$ translation: replace every subformula β by $\Box\beta$

Thus $p \to q$ may be *classically* interpreted as

$$\Box(\neg p \vee q)$$

Gregorczyk modal logic

Better Solution to VVOP (Grzegorczyk)

$$\alpha \in \textbf{Int} \iff \mathrm{Tr}(\alpha) \in \textbf{Grz}$$

Grz: normal extension of **S4** given by

$$\Box(\Box(p \to \Box p) \to p) \to p$$

Blok-Esakia theorem logically ('76)

The Best Solution to VVOP (Blok Esakia)

There is an isomorphism

$$\sigma : \mathrm{Ext}\,\mathbf{Int} \to \mathrm{NExt}\,\mathbf{Grz}.$$

s.t. for $\mathbf{L} \in \mathrm{Ext}\,\mathbf{Int}$

$$\alpha \in \mathbf{L} \iff \mathrm{Tr}(\alpha) \in \sigma(\mathbf{L})$$

Ext **Int** - lattice of extensions of **Int** (intermediate logics + trivial)

NExt **Grz** - lattice of normal extensions of **Grz**

algebra

Heyting algebras: distributive bounded lattices with residuation
$\mathcal{H}ey$: the variety of Heyting algebras

closure algebras: Boolean algebras with operation $\Box$ giving semantics for **S4**

Fact

There are one to one correspondences between

- extensions of **Int** and varieties of Heyting algebras
- normal extensions of **S4** and varieties of closure algebras

from closure algebras to Heyting algebras

$\mathbf{M}$ - closure algebas
$O(\mathbf{M}) = \{\Box p \mid p \in M\}$ - Heyting algebra of open elements of $\mathbf{M}$

$\mathcal{V}$ - variety of closure algebras
$\rho(\mathcal{V}) = \{O(\mathbf{M}) \mid \mathbf{M} \in \mathcal{V}\}$ - variety of Heyting algebras

from Heyting algebras to closure algebras

Theorem (McKinsey Tarski '46)

$\mathbf{H}$ - Heyting algebra
$\mathrm{B}(\mathbf{H})$ - *free Boolean extension* of $\mathbf{H}$

- $\mathrm{OB}(\mathbf{H}) = \mathbf{H}$;
- if $\mathbf{H} \leqslant \mathrm{O}(\mathbf{M})$, then $\mathrm{B}(\mathbf{H}) \cong \langle H \rangle_{\mathbf{M}}$
- every homomorphism $f : \mathbf{H} \to \mathrm{O}(\mathbf{M})$ extends uniquely to $\bar{f} : \mathrm{B}(\mathbf{H}) \to \mathbf{M}$

$\mathcal{V}$ - variety of Heyting algebras
$\sigma(\mathcal{V}) = \mathrm{HSP}\{\mathrm{B}(\mathbf{H}) \mid \mathbf{H} \in \mathrm{V}\}$

Blok-Esakia theorem algebraically

There mappings

$$\rho\colon \mathsf{L_V}(\mathcal{G}rz) \to \mathsf{L_V}(\mathcal{H}ey)$$
$$\sigma\colon \mathsf{L_V}(\mathcal{H}ey) \to \mathsf{L_V}(\mathcal{G}rz)$$

are mutually inverse lattice isomorphisms

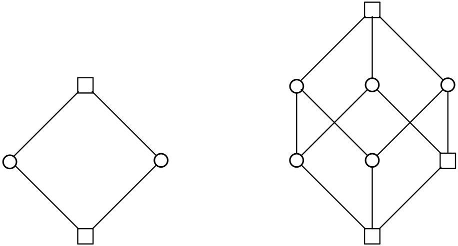
$\mathsf{L_V}(\mathcal{H}ey)$ - the lattice of varieties of Heyting algebras
$\mathsf{L_V}(\mathcal{G}rz)$ - the lattice of varieties of Grzegorczyk algebras

into the proof: What is it a Grzegorczyk algebra?

Proposition (Blok)

$\mathbf{M}$ - closure algebra

$\mathbf{M}$ is Grzegrorczyk iff $\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{M})$ and $\mathbf{S}_{1,2} \notin \mathsf{HS}(\mathbf{M})$



$\mathbf{S}_2$ $\quad$ $\mathbf{S}_{1,2}$

Corollary

Every $\mathsf{B}(\mathbf{H})$ is Grzegorczyk

into the proof: easy stuff?

Basic properties of O and B + McKinsey-Tarski theorem gives

Proposition

$\mathcal{V}$ - variety of Heyting algebras
$\mathcal{W}$ - variety of Grzegorczyk algebras Then

- $\rho\,\sigma(\mathcal{V}) = \mathcal{V}$
- $\sigma\rho\,(\mathcal{W}) \subseteq \mathcal{W}$

The lacking inclusion may be restated as

$$\mathcal{W} = \mathrm{HSP}\,\{\,\mathrm{BO}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{W}\,\}$$

and it follows from Blok's lemma

into the proof: Blok's lemma

Blok's lemma - baby version

$\mathbf{M}$ - finite Grzegorczyk algebra. Then $\mathbf{M} \cong \mathrm{BO}(\mathbf{M})$

into the proof: Blok's lemma

Blok's lemma - baby version

M - finite Grzegorczyk algebra. Then $\mathbf{M} \cong \mathrm{BO}(\mathbf{M})$

Blok's lemma

M - Grzegorczyk algebra

M embeds into some elementary extension of $\mathrm{BO}(\mathbf{M})$

into the proof: Blok's lemma

Blok's lemma - baby version

$\mathbf{M}$ - finite Grzegorczyk algebra. Then $\mathbf{M} \cong \mathrm{BO}(\mathbf{M})$

Blok's lemma

$\mathbf{M}$ - Grzegorczyk algebra
$\mathbf{M}$ embeds into some elementary extension of $\mathrm{BO}(\mathbf{M})$

Blok's lemma - detailed version

$\mathbf{M}$, $\mathbf{N}$ - Grzegorczyk algebras
$\mathbf{N} \leqslant \mathbf{M}$ and $\mathrm{O}(\mathbf{N}) = \mathrm{O}(\mathbf{M})$
$\varphi(x, \bar{y})$ - quantifier-free formula
$a \in (M - N)$, $\bar{b} \in N^n$

If $\mathbf{M} \models \varphi(a, \bar{b})$, then there is $c \in N$, $c \leqslant a$ such that

$$(\forall e \in M) \quad c \leqslant e \leqslant a \quad \Rightarrow \quad \mathbf{M} \models \varphi(e, \bar{b})$$

Elements from $\mathbf{M}$ may be "finitely approximated" in $\mathbf{N}$

Blok-Esakia theorem algebraically

What, it actually proves that $\mathcal{W} = \mathsf{SP_U}\{\,\mathsf{BO}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{W}\,\}$
not just $\mathcal{W} = \mathsf{HSP}\{\,\mathsf{BO}(\mathbf{M}) \mid \mathbf{M} \in \mathcal{W}\,\}$

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Theorem

There mappings

$$\rho\colon \mathsf{L_U}(\mathcal{G}rz) \to \mathsf{L_U}(\mathcal{H}ey)$$
$$\sigma\colon \mathsf{L_U}(\mathcal{H}ey) \to \mathsf{L_U}(\mathcal{G}rz)$$

are mutually inverse lattice isomorphisms

$\mathsf{L_U}(\mathcal{H}ey)$ - the lattice of universal subclasses of $\mathcal{H}ey$
$\mathsf{L_U}(\mathcal{G}rz)$ - the lattice of universal subclasses of $\mathcal{G}rz$

universal classes

universal sentences look like conjunctions of

$$(\forall\bar{x})\ s_1(\bar{x}) \approx t_1(\bar{t}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \rightarrow s'_1(\bar{x}) \approx t'_1(\bar{x}) \vee \cdots \vee s'_n(\bar{x}) \approx t'_n(\bar{x})$$

universal classes look like Mod(universal sentences)

These are classes closed under subalgebras and elementary equivalence

$\mathcal{V}$ - variety of Heyting algebras

$\sigma(\mathcal{V}) = \mathsf{SP_U}\{\mathsf{B}(\mathbf{H}) \mid \mathbf{H} \in \mathsf{V}\}$

ρ defined as previously

(multi-conclusion) deductive systems

$Sent$ - set of propositional sentences
Ax - axioms ($\subseteq Sent$)

$+$ inference rules: $\dfrac{\Delta}{\Sigma}, \quad \Delta, \Sigma \subseteq_{fin} Sent$

$+$ (some conditions)

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consequence relation $\vdash$

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consequence relation $\vdash$

Correspondence for algebraizable deductive systems

consequence relation $\vdash$	$\leftrightsquigarrow$	universal class $\mathcal{U}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities true in $\mathcal{U}$
derivable rules	$\leftrightsquigarrow$	universal sentences true in $\mathcal{U}$
single-conclusion der. rules	$\leftrightsquigarrow$	quasi-identities true in $\mathcal{U}$

Blok-Esakia theorem logically again

There is an isomorphism

$$\sigma : \text{DExt}\,\mathbf{Int} \to \text{DExt}\,\mathbf{Grz}.$$

Int - intuitionistic logic as a deductive system
DExt **Int** - lattice of its extensions

Grz - modal Grzegorczyk logic as a deductive system
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Reproved by E. Jeřabek in [*Canonical rules*, J. Symb. Log. 2005]

$\mathbf{Int}_{\boxtimes}$ and $\mathbf{Grz}_{\boxtimes}$

$\mathbf{Int}_{\boxtimes}$ - intuitionistic modal logic:
connectives: $0, 1, \wedge, \vee, \rightarrow, \boxtimes$
rules: modus ponens, necessitation
axioms: substitution closure of **Int**

$$+ \boxtimes\alpha \wedge \boxtimes\beta \leftrightarrow \boxtimes(\alpha \wedge \beta) + \boxtimes 1 \leftrightarrow 1$$

Semantics for $\mathbf{Int}_{\boxtimes}$

- algebraic: modal Heyting algebras
- relational: frames $(X, \leqslant, R)$ where $\leqslant \circ R \circ \leqslant = R$

$\mathbf{Int}_{\boxtimes}$ and $\mathbf{Grz}_{\boxtimes}$

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Semantics for $\mathbf{Int}_{\boxtimes}$

- algebraic: modal Heyting algebras
- relational: frames $(X, \leqslant, R)$ where $\leqslant \circ R \circ \leqslant \, = R$

$\mathbf{Grz}_{\boxtimes}$ - Grzegorczyk bimodal logic:
connectives: $0, 1, \wedge, \vee, \neg, \Box, \boxtimes$
rules: modus ponens, both necessitations
axioms: substitution closure of **Grz**

$$+ \ \boxtimes\alpha \wedge \boxtimes\beta \leftrightarrow \boxtimes(\alpha \wedge \beta) \ + \ \boxtimes 1 \leftrightarrow 1$$

$$+ \Box\boxtimes\Box\alpha \leftrightarrow \boxtimes\alpha$$

and again, Blok-Esakia theorem logically

There is an isomorphism

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$\mathbf{Int}_{\boxtimes}$ - intuitionistic modal logic as a deductive system
$\mathrm{DExt}\,\mathbf{Int}_{\boxtimes}$ - lattice of its extensions

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- the proof requires small additions to Blok's proof
- the part for logics (varieties) was proved by F. Wolter and M. Zakharyaschev in 1997
- what about diamond-like operations?

and again, Blok-Esakia theorem logically

There is an isomorphism

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- the proof requires small additions to Blok's proof
- the part for logics (varieties) was proved by F. Wolter and M. Zakharyaschev in 1997
- what about diamond-like operations? A naive approach gives modal companion but not a normal one. The existence of a normal one is OPEN!

For single-conclusion deductive systems or for quasivarieties

Studying free algebras and relative congruences gives:

Theorem

Let $\mathcal{P}$ be one of the properties

- admitting (parameterized, local) deduction theorem
- being (almost) structurally complete
- being finitely axiomatizable

Let $\mathbf{S}$ be a single-conclusion deductive system from DExt $\mathbf{Int}_{\boxtimes}$. Then $\mathbf{S}$ has $\mathcal{P}$ iff $\sigma(\mathbf{S})$ has $\mathcal{P}$.

preservation

For single-conclusion deductive systems or for quasivarieties

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Remark (Dziobiak, Rybakov)

There is a finite Heyting algebra $\mathbf{H}$ such that $\mathrm{Q}(\mathbf{H})$ is not finitely axiomatizable nor relative congruence distributive

The end

This is all

Thank you!