Optimal strong Mal'cev conditions for congruence meet-semidistributivity in locally finite varieties

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A strong Mal'cev condition Σ is *realized* in a variety $\mathcal{V}$ if there is an assignment of $\mathcal{V}$-terms to operation symbols of Σ such that the resulting identities become true in $\mathcal{V}$ (realization: weaker than interpretation, stronger than semantic embedding).

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Trivial observation: Σ_1 is realized in $\mathrm{Mod}(\Sigma_2)$ iff every variety which realizes Σ_2 realizes Σ_1. We denote this by $\Sigma_1 \preceq \Sigma_2$ and say Σ_1 is *weaker* than Σ_2.

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- *Assuming the language is finite, Baker's Single Sequence Lemma works (sort of...), so Park's conjecture can be proved in Baker's way.*
- *There exists some strong Mal'cev condition W_n among the family of Willard's conditions such that $\mathcal{V}$ realizes W_n.*
- *$\mathcal{V}$ realizes some idempotent linear Mal'cev condition which is not realized in any nontrivial variety of modules.*

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- *(JMMM)* $\mathcal{V}$ *realizes the strong Mal'cev condition* $t(x,x,x,y) \approx t(x,x,y,x) \approx t(x,y,x,x) \approx t(y,x,x,x) \approx t(x,x,y,y) \approx t(x,y,x,y) \approx t(x,y,y,x)$ *and* $t(x,x,x,x) \approx x$.

On the bounded width CSP

$(V, A, \mathcal{C})$ is an instance of the CSP where $\mathcal{C} = \{\langle C_1, W_1\rangle, \ldots, \langle C_m, W_m\rangle\}$, and $W_i \subseteq V$, while $C_i \subseteq A^{W_i}$. $f : V \to A$ is a solution of that instance if for all i, $f|_{W_i} \in C_i$.

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- (2-consistency) for all $u, v \in V$ and W_i, W_j such that $\{u, v\} \subseteq W_i \cap W_j$, $C_i|_{\{u,v\}} = C_j|_{\{u,v\}}$ and

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Theorem (Barto)

If $\mathbf{A}$ generates a congruence meet-semidistributive variety and $(V, A, \mathcal{C})$ is a $(2,3)$-minimal instance of CSP($\mathbf{A}$) such that all C_i are nonempty, then it has a solution.

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Then take $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. Let $(V, F, \mathcal{C})$ be the instance of CSP($\mathbf{A}$) which imposes on each triple $\{u, v, w\} \subseteq V$ the constraint

$$R_3 = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \end{bmatrix}\right)$$

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and on each 4-element subset of V the constraint

$$R_4 = \mathrm{Sg}^{\mathbf{F}^4}\left(\begin{bmatrix} x \\ x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \\ x \end{bmatrix}\right)$$

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$(V, F, \mathcal{C})$ is trivially 3-dense. It is 2-consistent because both R_3 and R_4 project to any pair of variables as

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$$\begin{bmatrix} c \\ c \\ c \end{bmatrix} \in R_3 \text{ and } \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} \in R_4.$$

which implies that there exist a ternary and a quarternary weak nu terms with derived binary operation $c(x, y)$. (QED)

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So we impose a ternary constraint forced by flavors on every triple of variables (there are 8 possibilities which arise), and also a 7-ary constraint on those septuples of variables which have the precise containment/disjointness/other relation to each other demanded by the desired equations.

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- *A_1 is disjoint from all others;*
- *$A_2, \ldots, A_7$ form a 3-crown poset under inclusion;*
- *Any incomparability that we see in that seven-element poset which can be disjointness, is disjointness;*
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Proof.

$|W_1| = 4$ and $|W_{n+1}| = 3(n+1)(2^{|W_n|} - 1) + 1.$ □

Another improvement of (KKVW)

Theorem

Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exists a binary term $t(x, y)$ and for all arities $n \geq 3$ terms $w_n(x_1, \ldots, x_n)$ such that

- *All w_n are weak near-unanimity terms in $\mathcal{V}$ and*
- *For all n, $\mathcal{V} \models w_n(x, x, \ldots, x, y) \approx t(x, y)$.*

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Proof.

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We pretty much convinced ourselves that any approach with CSP won't work.

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All our proofs are using the fact that a certain strong Mal'cev condition can be realized only in a trivial module variety (which is globally equivalent to $\mathrm{CSD}(\wedge)$). No idea if there are conditions which are weaker than $\mathrm{CSD}(\wedge)$ but collapse to it when restricted to locally finite varieties.

THANK YOU FOR YOUR ATTENTION!

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AND THANK YOU TO NEBOJŠA, MAJA AND THE ORGANIZING TEAM FOR EXCELLENT WORK!