The length of terms

Nebojša Mudrinski

Department of Mathematics and Informatics
University of Novi Sad

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Terms

Definition

Let **A** be an algebra on the language $\mathcal{L}$. For each $n \in \mathbb{N}$ and variables $x_1, \ldots, x_n$ we define the set of all terms with variables $x_1, \ldots, x_n$ in abbreviation $T(x_1, \ldots, x_n)$ as the smallest set with

1. $x_i \in T(x_1, \ldots, x_n)$ for each $i \in \{1, \ldots, n\}$;
2. if $t_1, \ldots, t_k \in T(x_1, \ldots, x_n)$ and $f \in \mathcal{L}$ of the length $k \in \mathbb{N}$ then $f(t_1, \ldots, t_k) \in T(x_1, \ldots, x_n)$.

Example

In the group $(\mathbb{Z}, +)$ we have $(x + y) + z \in T(x, y, z)$.

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Term operations

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Let $n \in \mathbb{N}$. With each term $t(x_1, \ldots, x_n)$ in the language of the algebra **A** we associate an n-ary term operation by interpretation of each operation symbol with corresponding operation in **A**. The set of all n-ary term functions of **A** we denote by $\mathrm{Clo}_n(\mathbf{A})$.

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Let $n \in \mathbb{N}$. The length of terms is a function $||\cdot|| : T(x_1,\ldots,x_n) \to \mathbb{N}$ such that:

1. $||x_1|| = \cdots = ||x_n|| = 1$;
2. if $k \in \mathbb{N}$, $f \in \mathcal{L}$ and $t_1,\ldots,t_k \in T(x_1,\ldots,x_n)$ such that $||t_1|| = n_1,\ldots,||t_k|| = n_k$ then $||f(t_1,\ldots,t_k)|| = 1 + n_1 + \cdots + n_k$.

Example

$||(x+y)+z|| = 1+||x+y||+||z|| = 1+(1+||x||+||y||)+||z|| = 5.$

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Term operations versus terms

Number of functions in finite algebras

If **A** is a finite algebra and $n \in \mathbb{N}$ then there is a finite number of distinct n-ary functions on the underlying set.

Remark

For each $n \in \mathbb{N}$, $T(x_1, \ldots, x_n)$ is an infinite set.

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Infinitely many different terms in $T(x_1, \ldots, x_n)$ represent the same n-ary term function, but only finitely many of them we need to represent all distinct n-ary term functions of the given algebra **A**.

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Circuit complexity problem

How to check?

Let $n \in \mathbb{N}$. Give an n-ary function on a finite algebra. Is it a term function (a circuit)?

When to stop?

A computer program can check all the n-ary terms starting from the smallest length, but when to stop? We should know the minimal length of terms such that all distinct term functions can be represented by terms of the length at most n.

The minimal length of terms

$$\gamma_{\mathbf{A}}(n) := \min\{m \in \mathbb{N} \mid (\forall f \in \mathrm{Clo}_n \mathbf{A})(\exists \mathtt{t})(||\mathtt{t}|| \leq m \wedge \mathtt{t}^{\mathbf{A}} = f)\}$$

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The Task

What we want to do?

Let $n \in \mathbb{N}$. Find $\gamma_{\mathbf{A}}(n)$ for different type of algebra **A**.

Theorem (G. Horváth and Ch. Nehaniv, 2014)

Let $n, k \in \mathbb{N}$ and **G** be a finite k-nilpotent group. Then $\gamma_{\mathbf{G}}(n) \leq c \cdot n^k$, where c is a constant that depends on **G**.

Remark

They have effectively calculated c as a function of $exp\mathbf{G}$ (exponent of a group).

Plan

We try to generalize this result and obtain bounds in some Mal'cev algebras.

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We try to generalize this result and obtain bounds in some Mal'cev algebras.

Mal'cev Algebras

Definition

Mal'cev term: $d(x,y,y) = d(y,y,x) = x$

Expanded groups

An algebra (A, F) is called an expanded group if there exist group operations in F.

Ω-groups

An expanded group $(A, +, -, 0, F)$ is called an Ω-group if for all $f \in F$ we have $f(0, \ldots, 0) = 0$.

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„Easy" expanded groups

Easy Ω-groups

Let $(V, +, -, 0)$ be a finite group with one additional unary operation $f : V \to V$ such that $f(0) = 0$. We are going to find $\gamma_{\mathbf{V}}(n)$, where $\mathbf{V} = (V, +, -, 0, f)$ and $n \in \mathbb{N}$.

Exponent

In $\mathbf{V}$ the group $(V, +, -, 0)$ has a finite exponent $expV$ because V is a finite set. We will denote $E = expV - 1$.

Repetition of the unary operation

Since V is a finite set there are $F \in \mathbb{N}$ and $k < F$ such that $f^{F+1} = f^k$. We choose the smallest such F.

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A bit more conditions

Remark

As G. Horváth and Ch. Nehaniv consider nilpotent groups we are going to consider that our easy expanded group $\mathbf{V}$ is supernilpotent.

Easy 3-supernilpoitent expanded group

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$ and let $n \in \mathbb{N}$.

Proposition

In a 3-supernilpotent expanded group $\mathbf{V}$, every two polynomials $p_1 \in \mathrm{Pol}_n\mathbf{V}$ and $p_2 \in \mathrm{Pol}_m\mathbf{V}$ which are absorbing at $(0, \ldots, 0)$ with value 0, and $n + m > 3$, mutually commute.

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Commutators and Higher Commutators

In Groups

If H, K are normal subgroups of a group **G** then $[H, K]$ is a normal subgroup generated by $\{[h, k] \mid h \in H,\ k \in K\}$, where $[h, k] := h^{-1}k^{-1}hk$ for all $h \in H$ and $k \in K$.

A. Bulatov, 2001

The term condition n-ary commutator $[\underbrace{\bullet, \dots, \bullet}_{n}]$ in a Mal'cev algebra is an n-ary operation on Con **A**.

Definition

Let **A** be an algebra and let $n \in \mathbb{N}$, $(a_1, \dots, a_n) \in A^n$, $a \in A$. An n-ary polynomial p is absorbing at $(a_1, \dots, a_n)$ with value a if $p(x_1, \dots, x_n) = a$ whenever there exists an $i \in \{1, \dots, n\}$ such that $x_i = a_i$.

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Absorbing Polynomials in Expanded Groups

Special case (E. Aichinger, N. M., 2010)

Let $n \in \mathbb{N}$. The *n-ary commutator* $[\underbrace{1, \ldots, 1}_{n}]$ of a Mal'cev algebra $\mathbf{A}$ is the congruence of $\mathbf{A}$ generated by $\{(p(a_1, \ldots, a_n), p(b_1, \ldots, b_n)) \mid a_1, \ldots, a_n, b_1, \ldots, b_n \in A, p$ is absorbing at $(a_1, \ldots, a_n)$ with value $p(a_1, \ldots, a_n)\}$.

Absorbing polynomials in expanded groups

Let $n \in \mathbb{N}$. An n-ary polynomial f of an expanded group $(V, +, -, 0, F)$ is *absorbing* if $f(a_1, \ldots, a_n) = 0$ whenever there exists an $i \in \{1, \ldots, n\}$ such that $a_i = 0$.

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3-Supernilpotent

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Mal'cev algebras that satisfy $[1, 1, 1, 1] = 0$ are called 3-supernilpotent.

Proposition (E. Aichinger and N. M., 2010)

Let **V** be a 3-supernilpotent expanded group. Then for every $k > 3$ and every k-ary polynomial of **V** which is absorbing is zero polynomial.

Proposition (E. Aichinger, M. Lazić and N. M., 2014)

In a 3-supernilpotent expanded group **V**, every ternary absorbing polynomial p is distributive with respect to $+$ to every component.

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The general idea

2-element Boolean algebras

Every function on a 2 element set can be represented as a term operation of 2-element Boolean algebras. Using logical operations $\wedge$, $\vee$ and $\neg$ in canonical disjunctive or conjunctive normal form we obtain a term that represents the given function on the two element set.

Nilpotent groups

G. Horváth and Ch. Nehaniv used a spacial form of terms that can be built by commutator of elements to count the largest possible length and from there obtained an upper bound.

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Distributors

Definition of d_k

For every $k \in \{1, \ldots, F\}$, we define functions $d_k : V^2 \to V$

$$d_k(x, y) = f^k(x + y) - f^k(y) - f^k(x),$$

for every $x, y \in V$.

Definition of d_k^L

For every $k \in \{1, \ldots, F\}$, we define functions $d_k^L : V^3 \to V$:

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Some Properties of Commutators and Distributors

Proposition

In **V**, $c(x,y) = -x - y + x + y$ is an absorbing polynomial.

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In **V**, d_k is an absorbing polynomial for every $k \in \{1, \ldots, F\}$.

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In **V**, for every $k \in \{1, \ldots, F\}$, d_k^L is absorbing polynomial and distributive operation with respect to $+$ to every component.

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In **V**, for every $k \in \{1, \ldots, F\}$, d_k^L is absorbing polynomial and distributive operation with respect to $+$ to every component.

Building up the terms

Variables with unary operation

Let
$$X := \{f^k(x_j) \mid k \in \{0, \dots, F\}, j \in \{1, \dots, n\}\},$$
where $f^0(x_j) := x_j$ for every $j \in \{1, \dots, n\}$,

Terms with commutators and distributors

$$D(X) := \{d_i(u, v) \mid i \in \{1, \dots, F\}, (u, v) \in X^2\},$$
$$C(X) := \{c(u, v) \mid (u, v) \in X^2\},$$

Let $D = (X \times D(X)) \cup (D(X) \times X)$ and
$C = (X \times C(X)) \cup (C(X) \times X)$.

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The Short Form

Theorem (E. Aichinger, M. Lazić and N. M., 2014)

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$ and let $n \in \mathbb{N}$. For every n-ary term function Φ on $\mathbf{V}$ there exist functions $\theta_i^L : X^3 \to \{0, 1, \ldots, E\}$, $\theta_i : X^2 \cup D \to \{0, 1, \ldots, E\}$, $\varepsilon : X^2 \cup C \cup D \to \{0, 1, \ldots, E\}$ for every $i \in \{1, \ldots, F\}$, numbers $\alpha_j \in \{0, 1, \ldots, E\}$ for every $j \in \{1, \ldots, n\}$, and numbers $\beta_j^i \in \{0, 1, \ldots, E\}$ for every $i \in \{0, 1, \ldots, F\}$ and $j \in \{1, \ldots, n\}$, such that the following is true

The Short Form

The short form of the terms

$$\begin{aligned}\Phi(x_1,\ldots,x_n) &= \sum_{i=1}^{F}\sum_{(u,v,w)\in X^3}\theta_i^L(u,v,w)d_i^L(u,v,w)\\ &+ \sum_{i=1}^{F}\sum_{(u,v)\in X^2\cup D}\theta_i(u,v)d_i(u,v)\\ &+ \sum_{i=0}^{F}\sum_{j=1}^{n}\beta_j^i f^i(x_j)\\ &+ \sum_{(u,v)\in X^2\cup C\cup D}\varepsilon(u,v)c(u,v)\end{aligned}$$

for all $x_1,\ldots,x_n$.

Lemmas for calculating the length

Lemma (E. Aichinger, M. Lazić and N. M., 2014)

Let $\mathbf{V} = (V, +, -, 0, F)$ be an expanded group and let $E = \exp(V, +, -, 0)$. If $t(x_1, \ldots, x_n)$, $n \in \mathbb{N}$ is a term function and $a \in \{0, \ldots, E\}$, then $||a\, t(x_1, \ldots, x_n)|| \leq E||t(x_1, \ldots, x_n)||$.

Lemma (E. Aichinger, M. Lazić and N. M., 2014)

1. $||d_i(u, v)|| \leq i + 4F + 9$;
2. $||d_i(d_j(u, v), w)|| = ||d_i(w, d_j(u, v))|| \leq 2i + 2j + 10F + 25$;
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The bound

Theorem (E. Aichinger, M. Lazić and N. M., 2014)

Let $\mathbf{V} = (V, +, -, 0, f)$ be a finite 3-supernilpotent expanded group, where f is a unary function such that $f(0) = 0$ and let $n \in \mathbb{N}$. Then $\gamma(n) \leq an^3 + bn^2 + cn + d$, where
$a = (F+1)^3(24EF^3 + 90EF^2 + 107EF + 4F^2 + 5F + 50E + 2)$,
$b = \frac{1}{2}(F+1)^2(9EF^2 + 27EF + 2F + 18E + 2)$,
$c = \frac{1}{2}(F+1)(EF + 2E + 2)$ and $d = -(F+1)$.

Thank You for the Attention!