Quasivariety of pseudo BCI-algebras and its properties

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A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,
(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,
(A3) $1 \to x = x$,
(A4) $1 \rightsquigarrow x = x$,
(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety (relatively 1-regular, relatively ideal determined, relatively congruence modular (arguesian), 1-conservative)

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Embeding into the $\{\rightarrow, \rightsquigarrow, 1\}$-reduct of residuated po-monoid

- Every pseudo-BCI-algebra is a $\{\rightarrow, \rightsquigarrow, 1\}$-subreduct of an semi-integral residuated po-monoid.

  Semi-integral residuated po-monoid:
  $(M, \leq, \cdot, \rightarrow, \rightsquigarrow, 1)$, where $(M, \cdot, 1)$ is a monoid, $\leq$ is a partial order on M, and $\rightarrow, \rightsquigarrow$ are binary operations on M satisfying the *residuation law*, for all $x, y, z \in M$:

  $$x \leq y \rightarrow z \quad \text{iff} \quad x \cdot y \leq z,$$

  $$x \leq y \rightsquigarrow z \quad \text{iff} \quad y \cdot x \leq z.$$

  The monoid identity 1 is a maximal element of the poset $(M, \leq)$.

In any pseudo BCI-algebra $(A, \to, \rightsquigarrow, 1)$ hold for all $x, y, z \in A$:

1. $x \to x = 1$, $x \rightsquigarrow x = 1$,
2. $x \rightsquigarrow ((x \to y) \rightsquigarrow y) = 1$, $x \to ((x \rightsquigarrow y) \to y) = 1$,
3. $x \to y = 1$ iff $x \rightsquigarrow y = 1$,
4. $x \leq y$ implies $y \to z \leq x \to z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
5. $x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
6. $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
7. $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,
8. $x \to y \leq (y \to x) \to 1$, $x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightsquigarrow 1$,
9. $x \to 1 = x \rightsquigarrow 1$,
10. $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \to 1)$,
    $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \rightsquigarrow 1)$,
11. $((x \to y) \rightsquigarrow y) \to y = x \to y$,
    $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$,
12. $(x \to y) \to ((z \to x) \to (z \to y)) = 1$,
    $(x \rightsquigarrow y) \rightsquigarrow ((z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)) = 1$,

Important subalgebras of pseudo-BCI-algebra $(A, \to, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \to 1) \to 1) \to x = x$

- Group part of A ... $G_A = \{a \to 1 \mid a \in A\}$

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Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ is a group in which 1 is the identity and $x^{-1} = x \to 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\to$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \to 1) \to 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

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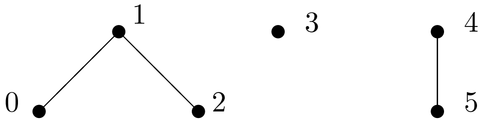
Example:

The set $A = \{0, 1, 2, 3, 4, 5\}$ equipped with the operations $\to$ and $\rightsquigarrow$ given by the following tables is a proper pseudo BCI-algebra:

$\to$	0	2	3	4	5	1
0	3	3	4	2	4	1
2	0	1	3	4	4	1
3	4	4	1	3	3	4
4	3	3	4	1	0	3
5	3	3	4	1	1	3
1	0	2	3	4	5	1

$\rightsquigarrow$	0	2	3	4	5	1
0	3	3	2	4	2	1
2	0	1	3	4	5	1
3	4	4	1	3	3	4
4	3	3	4	1	2	3
5	3	3	4	1	1	3
1	0	2	3	4	5	1

$(A, \leq)$



For the pseudo BCI-algebra we have $I_A = \{0, 1, 2\}$ and $G_A = \{1, 3, 4\}$ with the group operation $\cdot$ given by the following table:

$\cdot$	1	3	4
1	1	3	4
3	3	4	1
4	4	1	3

Given a pseudo-BCI-algebra $A = (A, \to, \rightsquigarrow, 1)$, the algebra $A^\star = (A, \rightsquigarrow, \to, 1)$ is a pseudo-BCI-algebra, too. A and $A^\star$ have the same underlying poset $(A, \leq)$, but it can easily happen that the algebras A and $A^\star$ are *not* isomorphic. For example, the "prelinearity identities"

$(x \to y) \to z \leq ((y \to x) \to z) \to z$ and

$(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$

are independent in general.

Group part $G_A^\star = (G_A, \rightsquigarrow, \to, 1)$:

$(G_A, \star)$, where $g \star h = h \cdot g$ for all $g, h \in G_A$, is a group isomorphic to $(G_A, \cdot)$ (isomorphism $g \mapsto g^{-1}$).

$G_A^\star = (G_A, \rightsquigarrow, \to, 1)$ ... $g \rightsquigarrow h = h \star g^{-1}$, $g \to h = g^{-1} \star h$

Relative congruences, filters and prefilters

Let $\mathscr{K}$ be a class of algebras of type F, $A \in \mathscr{K}$ and $\theta \in \mathrm{Con}(A)$.
We say that θ is a *relative congruence* (or *$\mathscr{K}$-congruence*) on A if $A/\theta \in \mathscr{K}$.

Denote $\mathrm{Con}_{\mathscr{K}}(A)$ the set of all relative congruences on A.

A prefilter in a pseudo BCI-algebra A is $D \subseteq A$ such that

(i) $1 \in D$,
(ii) if $x \in D$ and $x \to y \in D$, then $y \in D$,
(iii) for all $x \in A$, if $x \in D$ then $x \to 1 \in D$.

A prefilter D is a filter if, for all $x, y \in A$,

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Denote $\mathscr{F}(A)$ the set of all filters on A.
I_A is always a filter of A.

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Lemma

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra.

1. Any prefiltr is an order-filter, i.e., $x \in D$ and $x \leq y$ imply $y \in D$.
2. Any prefilter is a subalgebra of $(A, \to, \rightsquigarrow, 1)$.
3. $D \subseteq A$ is a prefilter if and only if $1 \in D$ and

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The filters correspond to the relative congruences: for every filter D,

$$\theta_D = \{(x, y) \mid x \to y \in D \text{ and } y \to x \in D\}$$

is the only relative congruence such that $[1]_{\theta_D} = D$.

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Remark

It is easy to show that

$$\theta = \{(x, y) \mid x \to 1 = y \to 1\}$$

is a congruence on A such that $[1]_\theta = I_A$, i.e. $\theta = \theta_{I_A}$.

Remark

The map $\alpha : a \mapsto a \to 1 = a \rightsquigarrow 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \rightsquigarrow, \to, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \rightsquigarrow, \to, 1)$.

The map $\beta : a \mapsto (a \to 1) \to 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \to, \rightsquigarrow, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \to, \rightsquigarrow, 1)$.

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Theorem

Let $(G, \cdot)$ be a group with the identity e and define $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$. Then $(G, \to, \rightsquigarrow, e)$ is a trivially ordered pseudo BCI- algebra where $\emptyset \neq H \subseteq G$ is a prefilter iff it is a subgroup of $(G, \cdot)$ and H is a filter iff it is a normal subgroup of $(G, \cdot)$.

Corollary

The lattice of prefilters need not be modular.
The lattice of filters need not be distributive.

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The direct product of I_A and G_A

For any $a, b \in A$ we define
$x \circ y = (x \to 1) \rightsquigarrow y$,
$x \star y = (y \rightsquigarrow 1) \to x$.

For the operations $\circ$ and $\star$ we have

1. $1 \circ x = x$, $x \star 1 = x$,
2. $x \circ (x \to 1) = 1$, $(x \rightsquigarrow 1) \star x = 1$,
3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$.

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3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$.

Lemma

For any pseudo-BCI-algebra A, the following are equivalent:

(1) The operation $\circ$ is associative;

(2) $(g \circ h) \circ x = g \circ (h \circ x)$ for all $g, h \in G_A$ and $x \in A$;

(3) $g^{-1} \rightsquigarrow (g \rightsquigarrow x) = x$ for all $g \in G_A$ and $x \in A$;

(4) The operation $\star$ is associative;

(5) $x \star (h \star g) = (x \star h) \star g$ for all $g, h \in G_A$ and $x \in A$;

(6) $g^{-1} \rightarrow (g \rightarrow x) = x$ for all $g \in G_A$ and $x \in A$.

Lemma

Let A be a pseudo-BCI-algebra. Then for all $x \in A$ and $g, h \in G_A$:

(i) $x \to g = (g \to x) \to 1$, $x \rightsquigarrow g = (g \rightsquigarrow x) \to 1$;

(ii) $g \to x = x \star g^{-1}$, $g \rightsquigarrow x = g^{-1} \circ x$;

(iii) $(x \to g) \to 1 = x \circ g^{-1}$, $(x \rightsquigarrow g) \to 1 = g^{-1} \star x$.

Proof.

We have $(g \to x) \to 1 = (g \to 1) \rightsquigarrow (x \to 1) = x \to ((g \to 1) \rightsquigarrow 1) = x \to g$, and similarly, $(g \rightsquigarrow x) \to 1 = x \rightsquigarrow g$.
Clearly, $x \star g^{-1} = (g^{-1} \rightsquigarrow 1) \to x = g \to x$,
$g^{-1} \circ x = (g^{-1} \to 1) \rightsquigarrow x = g \rightsquigarrow x$,
$x \circ g^{-1} = (x \to 1) \rightsquigarrow (g \to 1) = (x \to g) \to 1$ and
$g^{-1} \star x = (x \rightsquigarrow 1) \to (g \rightsquigarrow 1) = (x \rightsquigarrow g) \to 1$. □

Theorem

Let A be a pseudo-BCI-algebra. The following statements are equivalent:

(1) $A \cong I_A \times G_A \cong I_A \times {G_A}^{\star}$;

(2) G_A is a filter of A;

(3) A satisfies the equivalent conditions (1) - (6) of the Lemma and
$g \to x = g \rightsquigarrow x \quad$ for all $g \in G_A$, $x \in I_A$.

Properties of quasivariety of pseudo BCI-algebras

1) Quasivariety of pseudo BCI-algebras is relatively regular in 1

A quasi-variety $\mathscr{K}$ with a constant term 1 is said to be *relatively regular in 1*, if $[1]_\theta = [1]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \mathrm{Con}_{\mathscr{K}}(A)$.

It is known that a quasi-variety $\mathscr{K}$ is relatively regular in 1 iff there exist the terms $d_1(x, y), \ldots, d_n(x, y)$ in $\mathscr{K}$ such that $d_1(x, y) = 1, \ldots, d_n(x, y) = 1$ implies $x = y$.

Obviously, for the quasi-variety of all pseudo-BCI-algebras we can take $n = 2$, $d_1(x, y) = x \to y$ and $d_2(x, y) = y \to x$.

2) Quasivariety of pseudo BCI-algebras is relatively ideal determined

Let $\mathscr{K}$ be a class of algebras of type F with a constant 1. A term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of type F is called an <u>*ideal term*</u> in $y_1, \ldots, y_n$ if $\mathscr{K} \models t(x_1, \ldots, x_m, 1, \ldots, 1) = 1$.
A non-empty subset I of A is said to be <u>*closed under the ideal term*</u> $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $y_1, \ldots, y_n$ if $t(a_1, \ldots, a_m, b_1, \ldots, b_n) \in I$ whenever $b_1, \ldots, b_n \in I$.
We say that $\emptyset \neq I \subseteq A$ is an <u>*ideal*</u> in A if it is closed under all ideal terms.
The class $\mathscr{K}$ is called <u>*relatively ideal determined*</u> if for each $A \in \mathscr{K}$, every ideal in A is the kernel of a unique relative congruence on A.

Denote $\mathscr{I}(A)$ the set of all ideals on A.

Theorem

Let A be a pseudo BCI-algebra and $I \subseteq A$ with $1 \in I$. The following statements are equivalent:

(i) I is a filter of A.

(ii) $I = [1]_\theta$ for some $\theta \in \mathrm{Con}_{\mathscr{K}}(A)$.

(iii) I is an ideal of A.

(iv) I is closed with respect to the ideal terms

$$t_1(x_1, x_2, y_1, y_2) = (((y_1 \to (y_2 \to x_1)) \to x_1) \rightsquigarrow x_2) \rightsquigarrow x_2$$

$$t_2(y) = y \to 1$$

(v) I is closed with respect to the ideal terms

$$w_1(x, y_1, y_2) = (y_1 \to (y_2 \to x)) \to x$$

$$w_2(x, y) = (y \rightsquigarrow x) \rightsquigarrow x$$

$$t_2(y) = y \to 1$$

Corollary

$\mathrm{Con}_{\mathscr{K}}(A) \cong \mathscr{I}(A) = \mathscr{F}(A)$.

Let $\mathscr{K}$ be a relatively regular in 1 quasivarity in which there is a one-one correspondence between ideals and relative congruences, that is, for every algebra $A \in \mathscr{K}$, the map $\theta \mapsto [1]_\theta$ is an isomorphism of $\mathrm{Con}_{\mathscr{K}}(A)$ onto $\mathscr{I}(A)$. Then the following lemma holds:

Lemma

Let $\alpha, \beta \in \mathrm{Con}_{\mathscr{K}}(A)$. Then
$[1]_\alpha \vee [1]_\beta = \{a \in A \mid (a, b) \in \alpha$ for some $b \in [1]_\beta\} =$
$= \{a \in A \mid (a, b) \in \beta$ for some $b \in [1]_\alpha\}$, i.e. $a \in [1]_\alpha \vee [1]_\beta$ iff $(a, 1) \in \alpha \circ \beta$ iff $(a, 1) \in \beta \circ \alpha$.

Corollary

The lattices $\mathrm{Con}_{\mathscr{K}}(A) \cong \mathscr{I}(A) = \mathscr{F}(A)$ are modular.

3) Quasivariety of pseudo BCI-algebras is 1-conservative

A quasivariety $\mathscr{Q}$ with a constant 1 is said to be <u>*1-conservative*</u> if $\mathscr{Q}$ and the variety $\mathrm{HSP}(\mathscr{Q})$ generated by $\mathscr{Q}$ satisfy the same quasi-identities of the form

$$\bigwedge_{i=1}^{n} s_i(x_1, \ldots, x_k) = 1 \quad \Rightarrow \quad t(x_1, \ldots, x_k) = 1.$$

Proposition

Let $\mathscr{Q}$ be a relatively 1-regular quasivariety. Then $\mathscr{Q}$ is relatively ideal determined if and only if $\mathscr{Q}$ is 1-conservative and has a subtractive term (a binary term $s(x, y)$ such that $\mathscr{Q}$ satisfies the identities $s(x, x) = 1$ and $s(x, 1) = x$).

Corollary

Quasivariety of pseudo BCI-algebras is 1-conservative (substractive term $s(x, y) = y \to x$).

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THANK YOU
FOR YOUR ATTENTION!