Existential Pebble Games With Rank

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- Fixed template constraint satisfaction problem: essentially a homomorphism problem for finite relational structures.
- We are interested in membership in the class CSP($\mathbb{B}$), a computational problem that obviously lies in the complexity class **NP**.
- Dichotomy Conjecture (Feder and Vardi): either CSP($\mathbb{B}$) has polynomial time membership or it has **NP**-complete membership problem.

- We think of this problem as

$$\mathsf{HOM}(\mathbb{B}) = \{\mathbb{A} \mid \mathbb{A} \to \mathbb{B}\},$$

  where $\mathbb{A}$ ranges over finite structures in the fixed, finite signature of $\mathbb{B}$.
- We will approach this problem via descriptive complexity, i.e. try to capture its complexity via expressibility in an extension of FO logic which maintains poly-time model checking.
- Starting point: $\mathbb{B}$ is not provably intractable, i.e. $\mathbb{B}$ admits a weak near-unanimity polymorphism, idempotent polymorphisms.
- Goal: Show that, under certain additional assumptions, HOM($\mathbb{B}$) can be defined in the logic IFP+RK.

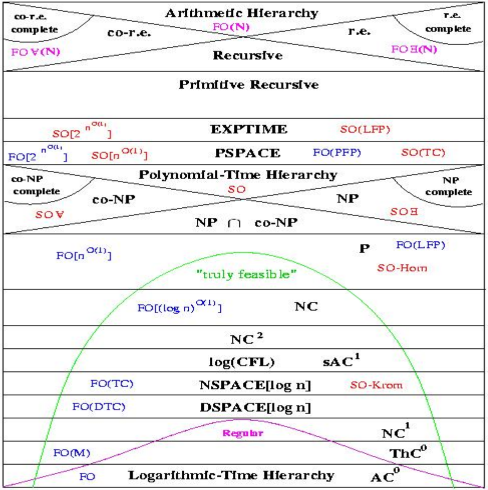


Figure: Descriptive Complexity Hierarchy

Existential k-Pebble Game

- Given: two similar relational structures $\mathbb{A}$ and $\mathbb{B}$, with $k \geq 1$ pairs of pebbles: $(a_1, b_1), \ldots, (a_k, b_k)$.
- Two players, the Spoiler and the Duplicator take turns in the following way: the Spoiler places k pebbles, one at a time, on elements of $\mathbb{A}$; the Duplicator responds by placing the matching pebbles on elements of $\mathbb{B}$.

- Once all (a_i, b_i), $1 \leq i \leq k$ have been placed, the Spoiler wins if one of the two conditions holds:
  1. the correspondence $a_i \mapsto b_i$, $1 \leq i \leq k$, is not a mapping;
  2. the correspondence $a_i \mapsto b_i$, $1 \leq i \leq k$, is a mapping but not a partial homomorphism from $\mathbb{A}$ to $\mathbb{B}$
- If none of the conditions hold, the Spoiler removes one or more pairs of pebbles and the game resumes.
- The Duplicator wins if she has a strategy that would enable her to play the game "forever".

- The structure $\mathbb{B}$ is of bounded width, if, there exists a $k \geq 1$ so that $\mathbb{A} \not\to \mathbb{B}$ is equivalent to the Spoiler having a winning strategy for the existential k-pebble game from $\mathbb{A}$ to $\mathbb{B}$.
- Algebraically, this amounts to $\mathbb{B}$ generating a congruence $\wedge$-semidistributive variety (Barto, Kozik, 2009)
- For us, it is interesting that this condition can be characterized in a fixed point logic.

Essentially, the strategy is the fixed point of the formula which is a disjunction of two statements:

1. the diagrams of the two induced substructures $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ disagree; and
2. $\bigvee_{i=1}^{k} (\exists x_i)\,(\forall y_i)($ "the configuration (x_i, y_i) leads to the winning strategy for the Spoiler").

over the predicate that captures the winning strategy for the Spoiler.

Limitations

- Existential pebble games fail to capture the class HOM($\mathbb{B}$), for structures which have the ability to count
- Roughly speaking, the ability to count indicate the presence of an Abelian group-like local structure lurking inside $\mathbb{B}$.
- Existential pebble games do not work since bipartite perfect matching can be associated to the ability to solve systems of linear equations over the group structure and this, in turn, requires very large monotone circuits violating bounded width.
- Question: is there a way to work around this issue and modify the game so that we add the ability to tackle problems closely associated to solving systems of linear equations over, say, finite fields $\mathbb{F}_p$?

Inflationary Fixed Point Logic (IFP)

- IFP: logic obtained from the first-order logic by closing it under formulas computing the inflationary fixed points of inductive definitions.
- On structures that come equipped with a linear order, IFP expresses precisely those properties that are in **P**.
- IFP cannot express evenness of a graph (pebble games.)

- Immermann: proposed IFP+C, a two-sorted extension of IFP with a mechanism that allows counting.
- There are existential quantifiers that count the number of elements of the structure which satisfy a formula φ. Also, we have a linear order built into one of the sort (essentially, positive integers.)
- FO quantifiers are bounded over the integer sort.
- There are polynomial time properties of digraphs not definable in IFP+C (Cai-Fürer-Immermann graphs; Bijection games)
- Atserias, Bulatov, Dawar (2007): IFP+C cannot express solvability of linear equations over $\mathbb{F}_2$.

Logic IFP+RK

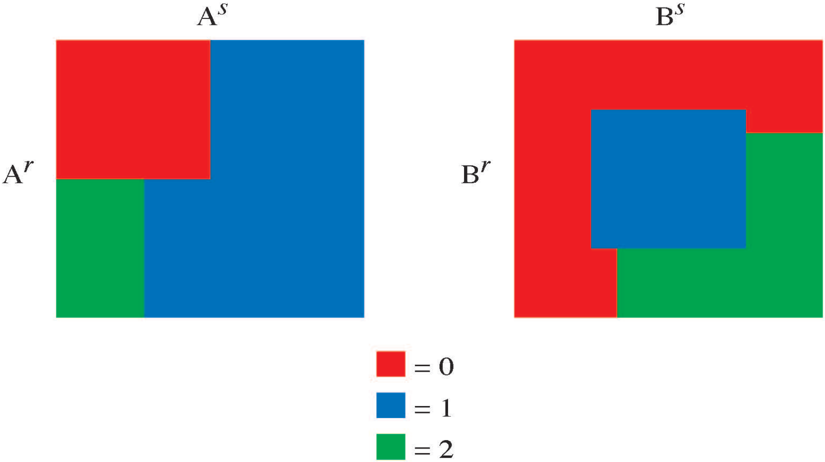
- IFP+RK is the logic obtained from IFP by adding the ability to compute the rank of a matrix over a finite field $\mathbb{F}_p$. It is a proper extension of IFP+C.
- Integer sort is equipped with the usual operations and relations $(+, \times, <)$; Quantifiers $\forall, \exists$ are still bounded over this sort.
- IFP+RK is poly-time testable on finite structures.
- All known examples of “natural” non-expressible properties in IFP+C can be handled in this logic. (Dawar, Grohe, Holm, Laubner)

Matrix Of A Logical Formula

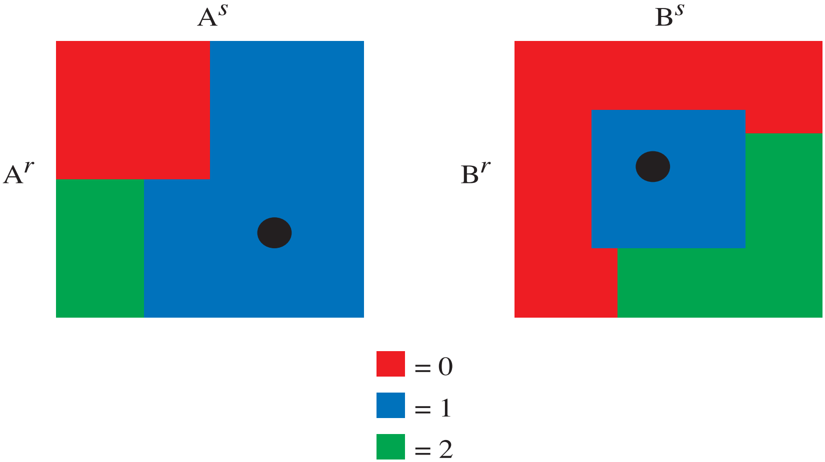
- Given a logical formula $\phi(\bar{x}, \bar{y})$ in the (r, s)-tuple of variables $(\bar{x}, \bar{y})$, it defines a $k \times l$-matrix $M_\phi(\bar{x}, \bar{y})$, where $k = |A^r|$ and $l = |A^s|$, with the entries in $\mathbb{F}_2$ in the obvious way.
- Given a finite collection of formulas $\Phi = \{\phi_1, \ldots, \phi_r\}$ in these same tuples of variables, we can obtain a matrix over $\mathbb{F}_p$, for a prime $p \geq r + 1$.
- We expand IFP by generalized quantifiers, which can compute the ranks of such matrices over any field $\mathbb{F}_p$ (p-prime)

k-Pebble m-Rank Partition Game

- Let $m \leq k$ and suppose we are given $\mathbb{A}$, $\mathbb{B}$.
- Spoiler starts by choosing an $l \leq m$ two positive integers r, s such that $r + s = l$.
- Spoiler then picks up l pebbles a_i from $\mathbb{A}$ and the corresponding l pebbles b_i from $\mathbb{B}$
- Duplicator responds by choosing the partition P of $\mathbb{A}^r \times \mathbb{A}^s$ and a partition Q of $\mathbb{B}^r \times \mathbb{B}^s$, which is rank-compatible
- Loosely speaking, what this means is that $|P| = |Q|$ and that corresponding blocks in both partitions are labelled by the same element of $\mathbb{F}_p$, and that the ranks of obtained matrices are identical
- Spoiler chooses a partition block of P then places $a_1, \ldots, a_l$ on an element of that block and places $b_1, \ldots, b_l$ on an element of the corresponding partition block of Q.



A^s
A^r
B^s
B^r
= 0
= 1
= 2


A^s
A^r
B^s
B^r
= 0
= 1
= 2

- Spoiler wins the game if, at some point, Duplicator cannot choose the right partitions or $\{(a_1, b_1), \ldots, (a_k, b_k)\}$ is not a partial isomorphism between $\mathbb{A}$ and $\mathbb{B}$.

Theorem

(Dawar, Holm, 2010) The k-pebble m-rank game over $\mathbb{F}_p$ captures the equivalence in the logic $\mathrm{IFP{+}R\kappa}^{k,m;p}$.

- Even though it is not entirely obvious, by choosing P and Q judiciously, this game subsumes the usual k-pebble game.
- We modify the k-pebble m-rank game by only allowing the matrices (partitions) arising from collections of formulas preserved under homomorphisms.
- The winning condition for Spoiler is either that the mapping is not a partial homomorphism or that Duplicator cannot pick the right rank partitions.

- The winning strategy for the Spoiler in the existential k-pebble partition game can be captured by an existential formula in IFP+RK with parameters (constants) from $\mathbb{B}$.
- We can think of such a formula as defining an "obstruction" to the existence of a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.
- In a way, think of that as capturing unsolvability in Datalog with negations and inequalities, which has the ability to invoke, at times, a basic linear system solver over finite fields.

Test Case: Conservative Problems

- Conservative structure: all subsets are distinguished by unary relations naming them.
- Algebraically: every nonempty subset is a subuniverse.

Theorem

(Bulatov [2003], Barto [2011]) The Dichotomy Conjecture holds for finite conservative templates.

Theorem

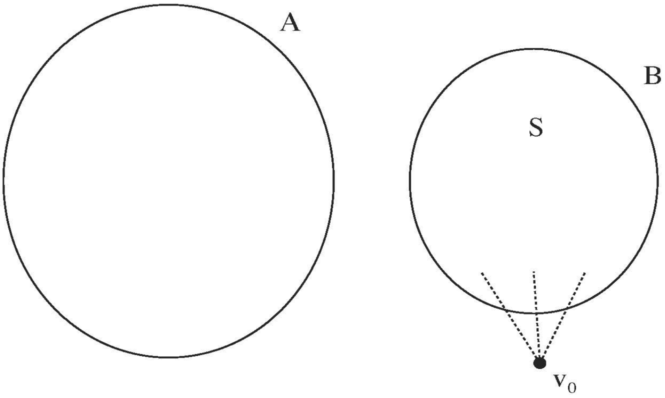
(D. [2015]) If $\mathbb{B}$ is a finite, conservative relational structure, HOM($\mathbb{B}$) *is expressible in the logic* IFP+RK.

- In fact, the rank function is only needed over the two-element field $\mathbb{F}_2$.
- The proof is inductive; it relies heavily on the fact that we can gradually extend minimal 2-element subuniverses to the full $\mathbb{B}$ using one-point extensions.

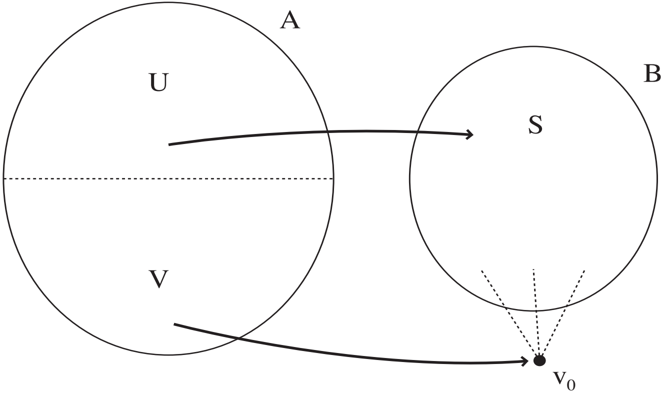
Sketch:

- The two-element subuniverses are term-minimal and idempotent, so they either (A. Szendrei)
  1. have a semilattice polymorphism;
  2. have a majority polymorphism;
  3. or are term equivalent to a 1-dimensional vector space over $\mathbb{F}_2$.
- In each of these cases, we have either the definability in Datalog or, in case (3), unsolvability is expressed by a formula stating that the ranks of two matrices do not coincide.

- Inductive step



- Inductive step



- The Spoiler's strategy for the extension can be pieced together from the strategies for S and $\{v_0\}$
- The substructures U (and V) of $\mathbb{A}$ can be defined by an inflationary operator, so U can be computed in time polynomial in the size of A.
- Various subproblems need to be solved, which check for the ability to map homomorphically certain definable subuniverses of U into subuniverses of S.

Obstacles to Generalizations:

- The property that every nonempty subset is a subuniverse is very strong.
- We need to partition B into subuniverses; one way to do it is to consider e.g. pp-definable equivalence relations of minimal index.
- In that case, we may be able to reduce the problem to the case where the algebraic structure is simple
- After that... ???

Problems

1. If $\mathbb{B}$ is a finite relational template with a weak near-unanimity polymorphism, is HOM($\mathbb{B}$) definable in IFP+RK?
2. If $\mathbb{B}$ is a Maltsev template, we know that HOM($\mathbb{B}$) is definable in IFP + RK (D., Habte, 2014+). Can this definability be obtained in FO+RK? (This is useful; it would place every such problem in the complexity class MOD_k, for some $k \geq 2$.