Algebra and the complexity of quantified constraints

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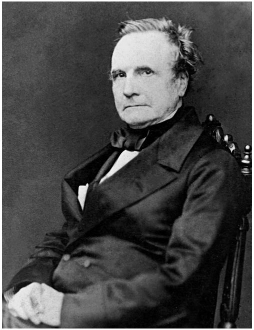
Mathematics



Computer Science

The Constraint Satisfaction Problem CSP($\mathcal{B}$) takes as input a *primitive positive* (pp) sentence Φ, i.e. of the form

$$\exists v_1 \ldots v_j \; \phi(v_1, \ldots, v_j),$$

where ϕ is a conjunction of atoms, and asks whether $\mathcal{B} \models \Phi$.

This is equivalent to the Homomorphism Problem – has $\mathcal{A}$ a homomorphism to $\mathcal{B}$?

The structure $\mathcal{B}$ is known as the template.

Finite CSPs occur a lot in nature.

- CSP($\mathcal{K}_m$) is graph m-colourability.
- CSP($\{0,1\}; R_{NAE}$), where B_{NAE} is $\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$ is not-all-equal 3-satisfiabilty.
- CSP($\{0,1\}; R_{TTT}, R_{TTF}, R_{TFF}, R_{FFF}$) is 3-satisfiabilty.
- CSP($\{0,1\}; \{0\}, \{1\}, \{(0,0),(1,1)\}$) is graph s-t unreachability.

Also vertex cover, clique and hamilton path – but these require non-fixed template.

Infinite CSPs also occur a lot in nature (another story...)

Feder-Vardi dichotomy conjecture. Each CSP($\mathcal{B}$) is either in P or is NP-complete.

- Compare with Ladner non-dichotomy for NP.

Still open, but known for:

- Structures size 2 (Schaefer 1978).
- Structures size 3 (Bulatov 2002).
- Structures with unary relations (Bulatov 2003).
- Smooth digraphs (Barto, Kozik and Niven 2010).
- Structures size 4 (Marković 2011?).

Manuel Bodirsky calls the CSP *Königsproblem* because it is a beautiful marriage of

- logic (primitive positive model theory)
- combinatorics (structure homomorphism)
- algebra (polymorphism clones and varieties)

to an important class of problems in computer science.

The Quantified CSP QCSP($\mathcal{B}$) takes as input a *positive Horn* (pH) sentence Φ, i.e. of the form

$$\forall \overline{v}_1 \exists \overline{v}_2 \ldots, Q\overline{v}_j \; \phi(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_j),$$

where ϕ is a conjunction of atoms, and asks whether $\mathcal{B} \models \Phi$.

QCSP($\mathcal{B}$) is always in Pspace.

- QCSPs used in AI to model non-monotonic reasoning.

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Previous classifications

QCSP classifications are harder than CSP classifications.

- Boolean structures. Dichotomy P, Pspace-complete. (Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.)
- Graphs of permutations. Trichotomy P, NP-complete, Pspace-complete. (Börner et al. 2002.)
- Various digraphs Dichotomies and trichotomies NL, NP-complete, Pspace-complete. (Madelaine, M. 2006, 2011, 2013; Dapić, Marković, M. 2014 etc.)
- Structures with 2-semilattice polymorphism. Dichotomy P, coNP-hard. (Chen 2004.)

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The recent advances in CSP complexity classification are due to the algebraic approach.

- a k-ary polymorphism of $\mathcal{B}$ is a homomorphism from $\mathcal{B}^k$ to $\mathcal{B}$.

The key to this approach is the Galois correspondence

$$\mathrm{Inv}(\mathrm{Pol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\mathrm{pp}}$$

whose consequence is

$$\mathrm{Pol}(\mathcal{B}) \subseteq \mathrm{Pol}(\mathcal{B}') \Rightarrow \mathrm{CSP}(\mathcal{B}') \leq_{\mathrm{P}} \mathrm{CSP}(\mathcal{B})$$

The algebraic approach exists also for the QCSP.

$$\mathrm{Inv}(\mathrm{sPol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\mathrm{pH}}$$

whose consequence is

$$\mathrm{sPol}(\mathcal{B}) \subseteq \mathrm{sPol}(\mathcal{B}') \Rightarrow \mathrm{QCSP}(\mathcal{B}') \leq_{\mathrm{P}} \mathrm{QCSP}(\mathcal{B}).$$

It appears to be weaker (surjective operations are not closed under composition) and we have fewer combinatorial constructs.

- Important?
- Königsproblem?

Assume henceforth that finite $\mathcal{B}$ contains constants naming each element. Now all polymorphisms are idempotent and $\mathrm{sPol}(\mathcal{B}) = \mathrm{Pol}(\mathcal{B})$.

Following Chen, for $z \in B$, call $\mathcal{B}$

- *logically* k*-collapsible* from source $\{z\}$

if truth of a pH sentence can be decided by sub-sentences in which all but k universal variables are forced to z.

E.g. $k := 2$ and for the pH sentence

$$\forall x_1 \forall x_2 \exists y_1 \forall x_3 \forall x_4 \exists y_2 \; E(x_1, y_1) \wedge E(x_2, y_1) \wedge E(x_3, y_2) \wedge E(x_4, y_2),$$

we obtain the 2-collapsings

$$\begin{array}{l}
\forall x_1 \forall x_2 \exists y_1 \exists y_2 \; E(x_1, y_1) \wedge E(x_2, y_1) \wedge E(z, y_2) \wedge E(z, y_2) \\
\forall x_1 \exists y_1 \forall x_3 \exists y_2 \; E(x_1, y_1) \wedge E(z, y_1) \wedge E(x_3, y_2) \wedge E(z, y_2) \\
\forall x_1 \exists y_1 \forall x_4 \exists y_2 \; E(x_1, y_1) \wedge E(z, y_1) \wedge E(z, y_2) \wedge E(x_4, y_2) \\
\forall x_2 \exists y_1 \forall x_3 \exists y_2 \; E(z, y_1) \wedge E(x_2, y_1) \wedge E(x_3, y_2) \wedge E(z, y_2) \\
\forall x_2 \exists y_1 \forall x_4 \exists y_2 \; E(z, y_1) \wedge E(x_2, y_1) \wedge E(z, y_2) \wedge E(x_4, y_2) \\
\exists y_1 \forall x_3 \forall x_5 \exists y_2 \; E(z, y_1) \wedge E(z, y_1) \wedge E(x_3, y_2) \wedge E(x_4, y_2)
\end{array}$$

If $\mathcal{B}$ is logically k-collapsible, then QCSP($\mathcal{B}$) "collapses" to an ensemble of instances of CSP($\mathcal{B}$) and QCSP($\mathcal{B}$) is in NP.

Call an idempotent clone $\mathbb{B}$

- *algebraically* *k-collapsible* from source $\{z\}$

if it contains f so that for each m, the image under f of set tuples that are co-ordinate permutations of

$$(\overbrace{B,\ldots,B}^{k \text{ times}},\overbrace{\{z\},\ldots,\{z\}}^{m-k \text{ times}})$$

is

$$(\overbrace{B,\ldots,B}^{m \text{ times}}).$$

E.g. $m := 4, k := 2$.

$$\begin{matrix} \{z\} & \{z\} & \{z\} & B & B & B & & B \\ \{z\} & B & B & \{z\} & \{z\} & B & f & B \\ B & \{z\} & B & \{z\} & B & \{z\} & \longrightarrow & B \\ B & B & \{z\} & B & \{z\} & \{z\} & & B \end{matrix}$$

Theorem (Chen 2006)

If Pol($\mathcal{B}$) is algebraically k-collapsible from source Z, then $\mathcal{B}$ is logically k-collapsible from source Z.

Theorem (Carvalho, Madelaine, M. 2015)

If $\mathcal{B}$ is logically k-collapsible from source Z, then Pol($\mathcal{B}$) is algebraically k-collapsible from source Z.

In many QCSP classiifications, all NP memberships can be explained uniformly by collapsibility. For example, this is true of all the classifications we already saw. But,

Theorem (Chen 2008)

There is $\mathcal{B}$ on 3-elements so that Pol($\mathcal{B}$) is "switchable" but not collapsible, and QCSP($\mathcal{B}$) is in NP.

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Collapsibility looks like a form of the polynomal generated powers property (PGP): E.g. $m := 4, k := 2$.

$$
\begin{matrix}
\{z\} & \{z\} & \{z\} & B & B & B & & B \\
\{z\} & B & B & \{z\} & \{z\} & B & f & B \\
B & \{z\} & B & \{z\} & B & \{z\} & \longrightarrow & B \\
B & B & \{z\} & B & \{z\} & \{z\} & & B
\end{matrix}
$$

Imagine for $|B| = 2$ that each column becomes

$$
\begin{matrix}
\{z\} & & z & z & z & z \\
\{z\} & \Rightarrow & z & z & z & z \\
B & \Rightarrow & b_1 & b_1 & b_2 & b_2 \\
B & & b_1 & b_2 & b_1 & b_2
\end{matrix}
$$

and for each $x_1, x_2, x_3, x_4 \in B$ there exists an f (i.e. this function is no longer uniform).

This is saying that B^m is generated, in Pol($\mathcal{B}$), from the set of tuples of the form of co-orindate permutations of $(b_1, \ldots, b_k, z, \ldots, z)$, a set that we call $\mathscr{C}^m_{\{z\}}$.

Theorem (Carvalho, Madelaine, M. 2015)

Pol($\mathcal{B}$) is algebraically k-collapsible from source Z iff, for all m, B^m is generated in Pol($\mathcal{B}$) by $\mathscr{C}^m_Z$.

Message: Collapsibility well understood in idempotent singleton source case; and quite well understood in general idempotent case.

Conjecture (Chen)

QCSP($\mathcal{B}$) in NP iff Pol($\mathcal{B}$) has the PGP; and Pspace-complete otherwise.