Some properties of the autocommutator subgroup of a 2−group

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Autocommutators. $\mathcal{N}-$autocommutators. The autocommutator subgroup

Definition

Let G be a group, $g \in G$ an element of it, and $\alpha \in Aut(G)$ an automorphism of G. The element $[g, \alpha] := g^{-1} \cdot g^{\alpha} \in G$ is called *the autocommutator of the element g with the automorphism α.*

Remark

The autocommutator $[g, \alpha]$ represents exactly the commutator in the holomorph $Hol(G)(:= G \rtimes Aut(G))$ of the elements g and α.

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Definition

Let G be a group and $\mathcal{N} \leq Aut(G)$ an automorphism group of G. The subgroup of G generated by all the autocommutators of the elements in G with the automorphisms in $\mathcal{N}$,

$$G_{\mathcal{N}} = [G, \mathcal{N}] = \langle [g, \nu] \mid g \in G, \nu \in \mathcal{N} \rangle$$

is called *the $\mathcal{N}-$autocommutator subgroup*. In particular, if $\mathcal{N} = Aut(G)$, the subgroup $G_{Aut(G)} =: K(G)$ is called *the autocommutator subgroup* of the group G.

Remark

Consider an arbitrary element $x \in G$ of the group G, and $i_x : G \longrightarrow G : g \longmapsto g^{i_x} = g^x = x^{-1}gx$ the inner automorphism associated to x. Then

$$[g, i_x] = g^{-1}g^x = [g, x] \quad , (\forall)g, x \in G \, .$$

Thus, in the case when $\mathcal{N} = Inn(G)$(the group of all inner automorphisms of the group G), we obtain that $G_{Inn(G)} = G'$.

The next result follows directly from the definition of $\mathcal{N}$−autocommutator subgroups.

Proposition

If $\mathcal{N}_1, \mathcal{N}_2 \leq Aut(G)$ are two automorphism groups of the group G, with $\mathcal{N}_1 \leq \mathcal{N}_2$, then $G_{\mathcal{N}_1} \leq G_{\mathcal{N}_2}$. In particular, $G' \leq K(G)$.

Taking **prop.1.5** into account, we deduce the following

Corollary

The autocommutator subgroup $K(G)$ is a normal subgroup of the group G. Also, the factor group $G/K(G)$ is commutative.

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Proposition

Let G be a group. Then $K(G)$ char G.

Remark

For $g \in G$ and $\alpha, \tau \in Aut(G)$ we also have

$$[g, \alpha]^{\tau} = (g^{-1} \cdot g^{\alpha})^{\tau} = g^{-\tau} \cdot g^{\alpha\tau} = (g^{\tau})^{-1} \cdot (g^{\tau})^{\tau^{-1}\alpha\tau} = [g^{\tau}, \alpha^{\tau}].$$

Hence, the image of an autocommutator through an automorphism is also an autocommutator. The set $AC(G)$ of all autocommutators of a group is thus invariant to the action of any automorphism, so that the autocommutator subgroup $K(G) = \langle AC(G) \rangle$ is characteristic in G.

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Remark

If G is a group, and we denote by $\mathcal{O}_x$ the orbit of an element $x \in G$ with respect to the natural action of the group $Aut(G)$, the autocommutator subgroup is given by

$$K(G) = \langle x^{-1}y |\, x, y \in G, (\exists)\alpha \in Aut(G) : y = x^{\alpha}\rangle =$$
$$= \langle x^{-1}y |\, x, y \in G, \mathcal{O}_x = \mathcal{O}_y\rangle = \langle x^{-1}y |\, (\exists)g \in G : x, y \in \mathcal{O}_g\rangle,$$

wherefrom we also immediately find that $K(G)\, char\ G$.

Remark

If G is a finite group of order $|G| > 2$, then $Aut(G) \neq \{id\}$, so that $K(G) \neq 1$.

Lemma

Let G be a group and H char G with $[G : H] = 2$. Then $K(G) \leq H$.

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Lemma

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Lemma

Let T be a group so that there are U, V char T such that $T = U \times V$. Then $K(T) = K(U) \times K(V)$.

Proposition

Let G and H two finite groups of coprime orders. Then $K(G \times H) = K(G) \times K(H)$.

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Let G and H two finite groups of coprime orders. Then $K(G \times H) = K(G) \times K(H)$.

Corollary

Let G be a finite nilpotent group. Then

$$K(G) = \prod_{p \in \pi(G)} K(P_p), \text{ where } Syl_p(G) = \{P_p\}.$$

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Let G be a finite nilpotent group. Then $K(G) = \prod_{p \in \pi(G)} K(P_p)$, where $Syl_p(G) = \{P_p\}$.

From the remark **1.10** and the corollary **1.14** we then immediately deduce

Corollary

Let G be a finite nilpotent group, $p \in \pi(G)$ and $Syl_p(G) = \{P\}$. If $p = 2$ and $P \ncong C_2$, or $p > 2$, then $p \in \pi(K(G))$.

Remark

In the conditions of the corollaries above, if $2 \in \pi(G)$, and $P_2 \cong C_2$ we find that $2 \not| \, |K(G)|$. This fact remains valid also for nonnilpotent groups.

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Proposition

Let G be a finite group of order $|G| = 4k + 2$, $k \in \mathbf{N}$. Then $|K(G)|$ is odd.

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Remark

A natural question one may raise is now whether for primes $p > 2$ the following implication holds

$$p \in \pi(G) \Longrightarrow p \in \pi(K(G)) \quad ?$$

Bearing **1.14** in mind, an eventual counterexample would be given by a nonnilpotent group.

Example

Let $G = \langle a, b | a^7 = b^3 = 1, a^b = a^2 \rangle$ be a noncommutative group of order 21. Then $G = \{a^k b^l | \, k = \overline{0,6}, l = \overline{0,2}\}$ and the orders of its elements are:

$$o(1) = 1$$
$$o(a^k) = 7, \ (\forall) k = \overline{1,6}$$
$$o(a^k b^l) = 3, \ (\forall) k = \overline{0,6}, l = \overline{1,2}$$

Since

$$(a^k)^{a^i b^j} = (a^k)^{b^j} = (a^k)^{2^j},$$

it follows that for any automorphism $\alpha \in Aut(G)$ the following hold

$$a^\alpha \in \{a^k \mid k = \overline{1,6}\},$$
$$b^\alpha \in \{a^i b \mid i = \overline{0,6}\}.$$

The orbits determined in G by the natural action of the group $Aut(G)$ of all of its automorphisms are then

$$\begin{aligned}
&\mathcal{O}_1 = \{1\}\,, \\
&\mathcal{O}_a = \{a, a^2, a^3, a^4, a^5, a^6\}\,, \\
&\mathcal{O}_b = \{b, ab, a^2b, a^3b, a^4b, a^5b, a^6b\}\,, \\
&\mathcal{O}_{b^2} = \{b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2\}\,.
\end{aligned}$$

We deduce then that $[g, \alpha] \in \langle a \rangle$, $(\forall) g \in G, \alpha \in Aut(G)$, so that $K(G) \subseteq \langle a \rangle$. Since $K(G) \neq 1$, it follows that $K(G) = \langle a \rangle$. Hence, even if $3 \mid |G|$, we see that $3 \nmid |K(G)|$.

Proposition

For any prime p there is a finite group G, so that $p \mid |G|$ but $p \nmid |K(G)|$.

A legitimate question one may raise is whether the autocommutator subgroup $K(G)$, generated by the autocommutators of the elements of the group G with the automorphisms in $Aut(G)$, consists only of autocommutators. It is known, in a similar context, that the commutator subgroup G' is not always formed just of commutators of the elements of the group G, even if some classes of groups are known for which G' consists only of commutators. The answer to this question is negative, as one can see from the next example:

Example

Let G be the group defined by the presentation

$$G = \langle a, b, c, d, e| \quad \begin{array}{l} a^4 = b^2 = c^2 = d^2 = e^2 = 1, \\ [a, x] = 1, [x, y] = a^2, (\forall) x, y \in \{b, c, d, e\} \rangle . \end{array}$$

The group G is a group of order $|G| = 64$, for which $G' = \langle a^2 \rangle = \{1, a^2\}$, $Z(G) = \langle a \rangle = \{1, a, a^2, a^3\}$, and G/G' is elementary abelian of order 32. Taking the definig relations into account, for any distinct $x, y, z, t \in \{b, c, d, e\}$ we have

$$(xy)^2 = (xyz)^2 = a^2 , \quad x^2 = (xyzt)^2 = 1$$

and we conclude immediately that

$$|g| = |a^2 g| \neq |ag| = |a^3 g| , (\forall) g \in G \setminus Z(G) .$$

We obtain that the $Aut(G)-$orbits in G are

$$1^{Aut(G)} = \{1\},$$
$$(a^2)^{Aut(G)} = \{a^2\},$$
$$a^{Aut(G)} = \{a, a^3\},$$
$$b^{Aut(G)} = \{x, a^2x, axy, a^3xy, axyz, a^3xyz, bcde, a^2bcde| \\ x, y, z \in \{b, c, d, e\}\},$$
$$(ab)^{Aut(G)} = \{ax, a^3x, xy, a^2xy, xyz, a^2xyz, abcde, a^3bcde| \\ x, y, z \in \{b, c, d, e\}\}.$$

The set $AC(G)$ of all autocommutators of the group G is then

$$AC(G) = \{1, a^2, x, ax, a^2x, a^3x, xy, axy, a^2xy, a^3xy, xyz, axyz, a^2xyz, a^3xyz, bcde, abcde, a^2bcde, a^3bcde \mid x, y, z \in \{b, c, d, e\}\} = G \setminus \{a, a^3\}.$$

Since $|AC(G)| = 62 > \frac{1}{2} \cdot 64 = \frac{1}{2}|G|$ it follows that $K(G) = \langle AC(G) \rangle = G$. Hence, the elements a and a^3 belong to the autocommutator subgroup $K(G)$, without being themselves autocommutators of some element $g \in G$ with some automorphism $\alpha \in Aut(G)$.
The set $AC(G)$ of all autocommutators of a group G is thus not necssarilly a subgroup of the group G, hence does not always coincide with the autocommutator subgroup $K(G)$.

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Groups with given autocommutator subgroup. Groups with the autocommutator subgroup a $p-$group

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In [2], P.Hegarty has proven the following result concerning the autocommutator subgroup of a group:

Proposition

Let K be a finite group. Then there are at most a finite number of finite groups G such that $K(G) \cong K$.

In his article, Hegarty obtains a limitation of the order of a group G for which $K(G) \cong K$, K being a given finite group.

In [3], M.Deaconescu and G.Walls determine all the groups with an subgroup autocommutator which is cyclic infinite or cyclic of prime order. Their results are the following:

Proposition

A group G has the property that $K(G) \cong \mathbb{Z}$ if and only if $G \cong \mathbb{Z}$, $G \cong \mathbb{Z} \times C_2$ or $G \cong \mathbb{Z} \rtimes C_2 = D_\infty$.

Proposition

Let p be a prime, and G a finite group so that $K(G) \cong C_p$.
a) If $p = 2$, then $G \cong C_4$.
b) If $p > 2$, then $G \cong C_p$, $G \cong C_p \times C_2$, $G \cong T$ or $G \cong T \times C_2$, where T is a partial holomorph of the cyclic group C_p, $T = C_p \rtimes \langle y \rangle$, cu $y \in Aut(C_p)$, so that $C_{\langle y \rangle}(C_p) = 1$.

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Some groups which are not autocommutator subgroups

Lemma

Let T be a group and $U \trianglelefteq T$ such that U is a complete group. Then $T = U \times C_T(U)$.

Proposition

Let G be a complete finite group such that $K(G) \neq G$. Then there is no finite group T such that $K(T) \cong G$.

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Example

If $n \in \mathbf{N}^*$, $n \neq 6$, the symmetric group S_n is complete and its autocommutator subgroup is $K(S_n) = A_n \neq S_n$. Hence there is no group G with $K(G) \cong S_n$.

On the other hand, there are important families of groups which can represent the autocommutator subgroup of a group.

Example

For any $n \in \mathbf{N}^*$, we have $K(S_n) \subseteq A_n = S_n' \subseteq K(S_n)$, so that $K(S_n) = A_n$.

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Finite abelian groups as autocommutator subgroups

Proposition

If G is a finite cyclic group, then $K(G) = G^2$.

Proposition

Let G be an abelian group of odd order. Then $K(G) = G$ and $K(G \times C_2) \cong G$.

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Proposition

Let G be a cyclic group of order 2^n, and H a $2-$group of exponent 2^m, with $m < n$. Then $K(G \times H) = G^2 \times H$.

Corollary

Let $n, m_1, m_2, \ldots, m_k \in \mathbf{N}^*$ with $n > m_1 \geq m_2 \geq \cdots \geq m_k$. Then $K(C_{2^n} \times C_{2^{m_1}} \times C_{2^{m_2}} \times \ldots \times C_{2^{m_k}}) = C_{2^{n-1}} \times C_{2^{m_1}} \times C_{2^{m_2}} \times \ldots \times C_{2^{m_k}}$. Hence any finite abelian $2-$group is the autocommutator subgroup of some finite group.

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Proposition

Any finite abelian group is isomorphic to the autocommutator subgroup of some finite group.

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Finite groups with the autocommutator subgroup a finite $p-$group

Proposition

Let p be a prime, and K a finite $p-$group. If G is a finite group with $K(G) = K$, then G has a unique $p-$Sylow subgroup P. Also, there is a an abelian $p'-$subgroup H of G, so that $G = P \rtimes H$.

Proposition

In the conditions of the previous proposition, if $H \leq Z(G)$, then
a) if $p = 2$, then $H = 1$.
b) if $p > 2$, then $H = 1$ or $H \cong C_2$.

Proposition

In the conditions of proposition 2.13, if $C = core_G(H)$, then $C = H \cap Z(G)$, and the abelian p'−group H/C has an exponent less or equal to $|K| - 1$.

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If the autocommutator subgroup K of a group G is a $p-$group of order p^m, then the order of the group G has no prime factors q with $q > p^m$.

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