Subdirectly irreducible commutative idempotent semirings

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Contents:

1. Semirings

Definition of a semiring and examples

Definition 1

A semiring is an algebra $(R, +, \cdot, 0, 1)$ *of type* $(2, 2, 0, 0)$ *satisfying*

- $(R, +, 0)$ *is a commutative monoid.*
- $(R, \cdot, 1)$ *is a monoid.*
- *The operation* $\cdot$ *is distributive with respect to* $+$.
- $x0 = 0x = 0$

Example 2

- $(\{0, 1, 2, 3, \ldots\}, +, \cdot, 0, 1)$ *is a semiring.*
- *Every unitary ring is a semiring.*
- *Every bounded distributive lattice is a semiring.*

2. Varieties of semirings

Different kinds of semirings

Definition 3

A semiring is called

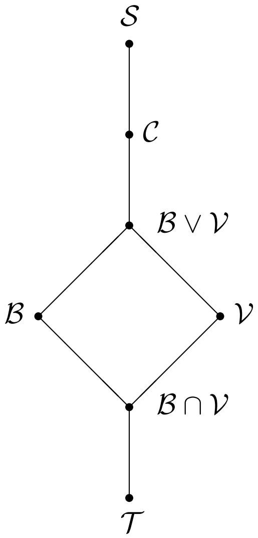
- *commutative if $\cdot$ is commutative*
- *idempotent if $\cdot$ is idempotent*
- *Boolean if it is commutative and idempotent and additionally satisfies* $1 + x + x = 1$
- *trivial if it has only one element*

Let

- $\mathcal{S}$ *denote the variety of semirings*
- $\mathcal{C}$ *the variety of commutative idempotent semirings*
- $\mathcal{B}$ *the variety of Boolean semirings*
- $\mathcal{V}$ *the subvariety of* $\mathcal{C}$ *determined by* $xy + x + 1 = x + 1$
- $\mathcal{T}$ *the variety of trivial semirings*

Hasse diagram of semiring varieties

We have the following Hasse diagram:



3. Subdirectly irreducible semrings

Subdirectly irreducible algebras

Definition 4

An algebra $\mathcal{A}$ with base set A is called subdirectly irreducible (SI) if there exists a smallest (with respect to $\subseteq$) congruence Θ on $\mathcal{A}$ with $\Theta \neq \Delta$ $(:= \{(x, x) \mid x \in A\})$, the so-called monolith of $\mathcal{A}$.

The importance of knowing all SI members of a variety is expressed by the following well-known fact:

Theorem 5

Every variety is generated by its SI members.

Therefore it is interesting to know all SI members of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{V}$.

Structure of SI members of $\mathcal{C}$

Lemma 6

If $(R,+,\cdot,0,1)\in\mathcal{C}$ *then* $(R,\cdot)$ *is a semilattice. We consider this semilattice as a meet-semilattice. Let* $\leq$ *denote the corresponding partial order relation. Then* $(R,\leq,0,1)$ *is a bounded poset.*

Lemma 7

If $\mathbf{R}=(R,+,\cdot,0,1)$ *is an SI member of* $\mathcal{C}$ *then there exists a coatom* a *of* $(R,\leq)$ *such that*

- $R=[0,a]\cup\{1\}$
- $\{a,1\}^2\cup\Delta$ *is the monolith of* $\mathbf{R}$.

Corollary 8

If $(R,+,\cdot,0,1)$ *is an SI member of* $\mathcal{C}$ *and* $|R|\leq 4$ *then* $(R,\leq)$ *is a chain.*

Definition of $\mathbf{S}_n$ and $\mathbf{T}_n$

Definition 9

For every integer $n > 1$ put $S_n := \{1, \ldots, n\}$ and let $\leq_1$ denote the linear ordering on S_n given by

$$\left.\begin{array}{l} 1 \leq_1 3 \leq_1 \ldots \leq_1 n-1 \leq_1 n \leq_1 n-2 \leq_1 \ldots \leq_1 2 \\ 1 \leq_1 3 \leq_1 \ldots \leq_1 n \leq_1 n-1 \leq_1 n-3 \leq_1 \ldots \leq_1 2 \end{array}\right\} \text{ if } n \text{ is } \left\{\begin{array}{l} \text{even} \\ \text{odd} \end{array}\right.$$

Moreover, put

$$\begin{aligned} x +_1 y &:= \max{}_{\leq_1}(x, y) \\ x +_2 y &:= \begin{cases} n-1 & \text{if } x = y = n \\ x +_1 y & \text{otherwise} \end{cases} \\ xy &:= \min(x, y) \\ \mathbf{S}_n &:= (S_n, +_1, \cdot, 1, n) \\ \mathbf{T}_n &:= (S_n, +_2, \cdot, 1, n) \end{aligned}$$

Definition of $\mathbf{S_C}$ and $\mathbf{T_C}$

Definition 10

For every infinite bounded chain $\mathbf{C} = (C, \leq_2, 0, 1)$ *let* $\mathbf{S_C}$ *denote the algebra* $(S_{\mathbf{C}}, +, \cdot, (0,1), (1,2))$ *of type* $(2,2,0,0)$ *defined by* $S_{\mathbf{C}} := C \times \{1,2\}$,

$$(x,i)+(y,j) := \left\{\begin{array}{l} (\max_{\leq_2}(x,y),1) \\ (y,2) \\ (x,2) \\ (\min_{\leq_2}(x,y),2) \end{array}\right\} \text{ if } (i,j) = \left\{\begin{array}{l} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array}\right.$$

$$(x,i)(y,j) := \left\{\begin{array}{l} (x,i) \\ (x,\min(i,j)) \\ (y,j) \end{array}\right\} \text{ if } x \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} y$$

$((x,i),(y,j) \in S_{\mathbf{C}})$. *Moreover, let* $\mathbf{T_C}$ *denote the algebra of type* $(2,2,0,0)$ *which coincides with* $\mathbf{S_C}$ *with the only exception that* $(1,2)+(1,2) := (1,1)$ *instead of* $(1,2)+(1,2) := (1,2)$.

Definition of $\mathbf{B} \oplus 1$

Definition 11

For any non-trivial Boolean lattice $\mathbf{B} = (B, \vee, \wedge, 0, a)$ *let* $\mathbf{B} \oplus 1$ *denote the semiring* $(S, +, \cdot, 0, 1)$ *where* $1 \notin B$, $S := B \cup \{1\}$ *and* $+$ *and* $\cdot$ *are defined as follows:*

$$x + y \;:=\; \left\{ \begin{array}{l} x \vee y \\ 1 \\ a \end{array} \right\} \text{ if } \left\{ \begin{array}{l} x, y \neq 1 \\ (x, y) \in \{(0, 1), (1, 0)\} \\ \textit{otherwise} \end{array} \right.$$

$$xy \;:=\; \left\{ \begin{array}{l} x \wedge y \\ y \\ x \end{array} \right\} \text{ if } \left\{ \begin{array}{l} x, y \neq 1 \\ x = 1 \\ y = 1 \end{array} \right.$$

SI semirings

Theorem 12

- *(Guzmán 92) Up to isomorphism* $\mathbf{S}_2$ *and* $\mathbf{T}_2$ *are all SI members of* $\mathcal{B}$.
- $\mathbf{S}_n$, $\mathbf{T}_n$, $\mathbf{S_C}$, $\mathbf{T_C}$ *and* $\mathbf{B} \oplus 1$ *are SI members of* $\mathcal{C}$.
- $\mathbf{S}_2$, $\mathbf{T}_3$ *and* $\mathbf{B} \oplus 1$ *are SI members of* $\mathcal{V}$.

Remark 13

- *If* $n \leq 4$ *then up to isomorphism* $\mathbf{S}_n$ *and* $\mathbf{T}_n$ *are the only n-element SI members of* $\mathcal{C}$.
- *If* $\mathbf{C}$ *is an n-element chain then* $\mathbf{S_C} \cong \mathbf{S}_{2n}$ *and* $\mathbf{T_C} \cong \mathbf{T}_{2n}$.
- $\mathbf{T}_3 \cong \mathbf{2} \oplus 1$
- $\mathbf{S}_n \in \mathcal{V}$ *if and only if* $n = 2$
- $\mathbf{T}_n \in \mathcal{V}$ *if and only if* $n = 3$
- $\mathbf{S_C}, \mathbf{T_C} \notin \mathcal{V}$

4. Concluding remarks

Number of SI semirings

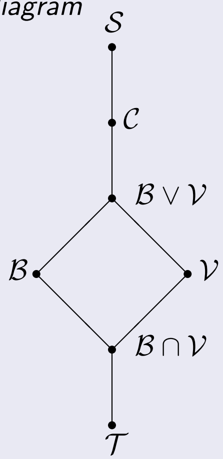
Remark 14

- *Up to isomorphism $\mathcal{B}$ has exactly two SI members.*
- *Up to isomorphism $\mathcal{C}$ has at least two SI members of cardinality $n > 1$.*
- *If $m \geq 2$ then up to isomorphism $\mathcal{C}$ has at least three SI members of cardinality $2^m + 1$.*
- *If $m \geq 0$ then up to isomorphism $\mathcal{V}$ has at least one SI member of cardinality $2^m + 1$.*
- *Since for infinite n there exist 2^n pairwise non-isomorphic Boolean algebras of cardinality n, $\mathcal{V}$ has at least 2^n SI members of infinite cardinality n.*

Hasse diagram of semiring varieties revisited

Remark 15

All inclusions in the Hasse diagram



are proper.

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Thank you for your attention!