Fraïssé categories and their applications

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Motivations

- Classical Fraïssé theory

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- More recent works

M. Droste, R. Göbel, *A categorical theorem on universal objects and its application in abelian group theory and computer science*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), 49–74, Contemp. Math., 131, Part 3, Amer. Math. Soc., 1992.

T. Irwin, S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. **358**, no. 7 (2006) 3077–3096.

W. Kubiś, S. Solecki, *A proof of uniqueness of the Gurarii space*, Israel J. Math. 195 (2013) 449–456.

The setup

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Assumptions on $\langle \mathfrak{S}, \mathfrak{L} \rangle$:

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(A2) For every $X = \lim \vec{x} \in \mathrm{Obj}(\mathfrak{L})$, $y \in \mathrm{Obj}(\mathfrak{S})$, for every arrow $f \colon y \to X$ there exists n such that $f = x_n^\infty \circ f'$ for some $f' \in \mathfrak{S}$.

Remark

For every category $\mathfrak{S}$ there exists a category $\sigma\mathfrak{S}$ such that $\langle \mathfrak{S}, \sigma\mathfrak{S} \rangle$ satisfies (A1), (A2).

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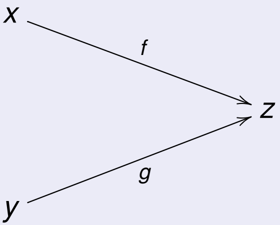
- The objects of $\sigma\mathfrak{S}$ are sequences (i.e., covariant functors) of type $\mathbb{N} \to \mathfrak{S}$.
- The $\sigma\mathfrak{S}$-arrows are natural transformations into subsequences.

Definition

We say that $\mathfrak{S}$ is directed if for every $x, y \in \mathrm{Obj}(\mathfrak{S})$ there exist $z \in \mathrm{Obj}(\mathfrak{S})$ and $\mathfrak{S}$-arrows $f \colon x \to z$, $g \colon y \to z$.



Definition

We say that $\mathfrak{S}$ has the amalgamation property if for every $\mathfrak{S}$-arrows $f\colon z \to x$, $g\colon z \to y$ there exist $\mathfrak{S}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that the diagram

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ z & \xrightarrow[f]{} & x \end{array}$$

is commutative.

Domination

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Definition

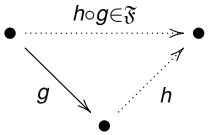
Let $\mathfrak{F}$ be a subcategory of $\mathfrak{S}$. We say that $\mathfrak{F}$ is dominating in $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $f \colon x \to y$ such that $y \in \mathrm{Obj}(\mathfrak{F})$.

(D2) Given an $\mathfrak{S}$-arrow g with $\mathrm{dom}(g) \in \mathrm{Obj}(\mathfrak{F})$, there exists an $\mathfrak{S}$-arrow h such that $h \circ g \in \mathfrak{F}$.

(D1) $x \dashrightarrow y \in \mathrm{Obj}(\mathfrak{F})$

(D2)



Main definition

Definition

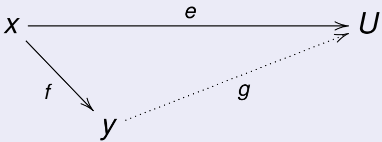
We say that $\mathfrak{S}$ is a Fraïssé category if

- $\mathfrak{S}$ is directed,
- $\mathfrak{S}$ has the amalgamation property,
- $\mathfrak{S}$ is dominated by a countable subcategory.

Theorem (Droste & Göbel 1993)

Assume $\mathfrak{S}$ is a Fraïssé category. Then there exists a unique, up to isomorphism, object $U \in \mathrm{Obj}(\mathfrak{L})$ with the following properties:

1. *For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{L}$-arrow $e \colon x \to U$.*
2. *For every $e \colon x \to U$ with $x \in \mathrm{Obj}(\mathfrak{S})$, for every $\mathfrak{S}$-arrow $f \colon x \to y$ there exists an $\mathfrak{L}$-arrow $g \colon y \to U$ such that $e = g \circ f$.*



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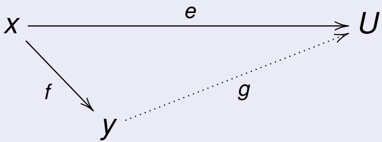

Definition

We call U the Fraïssé limit of $\mathfrak{S}$ and write $U = \mathrm{Flim}(\mathfrak{S})$.

Important features of Fraïssé limits

Theorem (Universality)

Let $U = \mathrm{Flim}(\mathfrak{S})$*. Then for every* $X \in \mathrm{Obj}(\mathfrak{L})$ *there exists an* $\mathfrak{L}$*-arrow* $e\colon X \to U$*.*

Important features of Fraïssé limits

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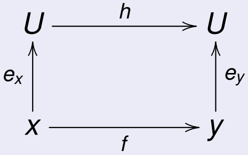
Let $U = \mathrm{Flim}(\mathfrak{S})$*. Then for every* $X \in \mathrm{Obj}(\mathfrak{L})$ *there exists an* $\mathfrak{L}$*-arrow* $e\colon X \to U$*.*

Theorem (Homogeneity)

Let $U = \mathrm{Flim}(\mathfrak{S})$*. For every* $\mathfrak{S}$*-arrow* $f\colon x \to y$*, for every* $\mathfrak{L}$*-arrows* $e_x\colon x \to U$*,* $e_y\colon y \to U$ *there exists an automorphism* $h\colon U \to U$ *satisfying* $h \circ e_x = e_y \circ f$*.*



Some examples

Example

Let $\mathfrak{S}$ be a category of finitely generated models of a fixed first-order language, $\mathfrak{L}$ a suitable category of countably generated structures. If $\mathfrak{S}$ is hereditary, then $\mathrm{Flim}(\mathfrak{S})$ is the same as the Fraïssé limit in the model-theoretic sense.

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Let $\mathfrak{S}$ be the category of nonempty finite sets, where $\mathfrak{S}(x, y)$ consists of all surjections from y onto x.
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Let $\mathfrak{S}$ be the category of nonempty finite sets, where $\mathfrak{S}(x, y)$ consists of all surjections from y onto x.
Let $\mathfrak{L}$ be the category of nonempty compact metrizable totally disconnected spaces with continuous surjections (again the arrows are reversed).
Then $\operatorname{Flim}(\mathfrak{S})$ is the Cantor set.

Monoids

Example

Let $\langle S, \circ \rangle$ be a monoid (i.e. a semigroup with a unit), viewed as a category. It is automatically directed. Amalgamation means

$$(\forall\, x, y \in S)(\exists\, x', y' \in S)\;\; x' \circ x = y' \circ y.$$

This holds, for example, when $\langle S, \circ \rangle$ is commutative.
Note that

$$T \subseteq S \text{ is dominating} \iff (\forall\, x \in S)(\exists\, y \in S)\;\; y \circ x \in T.$$

Metric spaces

Example

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Let $\mathfrak{S}$ be the category of finite metric spaces with isometric embeddings. Then $\mathfrak{S}$ is directed and has the amalgamation property. Unfortunately, $\mathfrak{S}$ is not countably dominated.

On the other hand, there exists a unique complete separable metric space $\mathbb{U}$, called the Urysohn space, with the following properties:

1. $\mathbb{U}$ contains isometric copies of all finite metric spaces.
2. Every isometry between finite subsets of $\mathbb{U}$ extends to a bijective isometry of $\mathbb{U}$.

So, $\mathbb{U}$ behaves like the Fraïssé limit of $\mathfrak{S}$. How to deal with it?
Note that if $\mathfrak{L}$ is the category of complete separable metric spaces then the pair $\langle \mathfrak{S}, \mathfrak{L} \rangle$ satisfies (A1) but it fails (A2).

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- Eve starts the game by choosing $u_0 \in \mathrm{Obj}(\mathfrak{S})$.
- Odd responds by choosing an $\mathfrak{S}$-arrow $u_0^1 \colon u_0 \to u_1$.

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- Eve starts the game by choosing $u_0 \in \mathrm{Obj}(\mathfrak{S})$.
- Odd responds by choosing an $\mathfrak{S}$-arrow $u_0^1 \colon u_0 \to u_1$.
- Eve responds by choosing an $\mathfrak{S}$-arrow $u_1^2 \colon u_1 \to u_2$.

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- Eve starts the game by choosing $u_0 \in \mathrm{Obj}(\mathfrak{S})$.
- Odd responds by choosing an $\mathfrak{S}$-arrow $u_0^1 \colon u_0 \to u_1$.
- Eve responds by choosing an $\mathfrak{S}$-arrow $u_1^2 \colon u_1 \to u_2$.
- And so on...

The result is a sequence $\vec{u}$:

$$u_0 \xrightarrow{u_0^1} u_1 \xrightarrow{u_1^2} u_2 \xrightarrow{u_2^3} u_3 \longrightarrow \cdots$$

We say that Odd wins if U is isomorphic to $\lim \vec{u}$. Otherwise Eve wins.

Generic objects

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Proof.

Supposing there are two generic objects and Odd uses his strategy for the first one, Eve can play using Odd's strategy for the second one. □

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Claim

$\mathfrak{S}$ *from the above example has a dominating Fraïssé subcategory.*

Question

Assume $\mathfrak{S}$ is countable, $U \in \mathrm{Obj}(\mathfrak{L})$, and Odd has a winning strategy in $\mathrm{BM}(\mathfrak{S}, U)$.
Does $\mathfrak{S}$ contain a subcategory with the amalgamation property?

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Under the assumptions above, $\mathfrak{S}$ is directed.

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Proof.

Eve can start the game with an arbitrary $\mathfrak{S}$-object x, showing that there is an $\mathfrak{L}$-arrow $f_x \colon x \to U$.
Taking another $\mathfrak{S}$-object y, we get $f_y \colon y \to U$.
Using (A2), we find m, n such that $f_x = u_m^\infty \circ g_x$ and $f_y = u_n^\infty \circ g_y$ for some $\mathfrak{S}$-arrows g_x, g_y.
Without loss of generality, $n = m$, showing that $\mathfrak{S}$ is directed. □

Metric spaces again

Theorem

Let $\mathfrak{S}$ be the category of finite metric spaces and let $\mathfrak{L}$ be the category of complete separable metric spaces, both with isometric embeddings. Then Odd has a winning strategy in $\mathrm{BM}(\mathfrak{S}, \mathbb{U})$*, where $\mathbb{U}$ is the Urysohn space.*

Banach spaces

Theorem

Let $\mathfrak{S}$ *be the category of finite-dimensional Banach spaces and let* $\mathfrak{L}$ *be the category of separable Banach spaces, both with linear isometric embeddings.*
Then there exists $\mathbb{G} \in \mathrm{Obj}(\mathfrak{L})$ *such that Odd has a winning strategy in* $\mathrm{BM}(\mathfrak{S}, \mathbb{G})$.

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The Banach space $\mathbb{G}$ is known, it is called the Gurariĭ space.
It was constructed by Gurariĭ in 1966.
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Remark

The Gurariĭ space $\mathbb{G}$ is not homogeneous, however every linear isometry between its finite-dimensional subspaces can be approximated by bijective linear isometries of $\mathbb{G}$.

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$\mathfrak{L}$ is enriched over the category of metric spaces with non-expansive mappings.
This means that each hom-set $\mathfrak{L}(X, Y)$ has a metric $\varrho = \varrho_{X,Y}$ such that

1. $\varrho(f \circ g_1, f \circ g_2) \leqslant \varrho(g_1, g_2)$
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(A2) If $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{S}$, then for every $\mathfrak{L}$-arrow $f \colon y \to X$, for every $\varepsilon > 0$ there exist n and an $\mathfrak{S}$-arrow $f' \colon y \to x_n$ such that $\varrho(x_n^\infty \circ f', f) < \varepsilon$.

Domination revisited

Definition

Let $\mathfrak{F}$ be a subcategory of $\mathfrak{S}$. We say that $\mathfrak{F}$ is dominating in $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $f \colon x \to y$ such that $y \in \mathrm{Obj}(\mathfrak{F})$.

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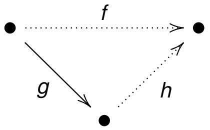

Definition

We say that $\mathfrak{S}$ has the almost amalgamation property if for every $\mathfrak{S}$-arrows $f \colon z \to x$, $g \colon z \to y$, for every $\varepsilon > 0$ there are $\mathfrak{S}$-arrows $f' \colon x \to w$, $g' \colon y \to w$ such that

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Definition

We say that $\mathfrak{S}$ is a Fraïssé category if it is directed, countably dominated and has the almost amalgamation property.

Theorem

Let $\mathfrak{S}$ be a Fraïssé category. There exists a unique, up to isomorphism, $\mathfrak{L}$-object U satisfying

1. *For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{L}$-arrow $e\colon x \to U$.*
2. *For every $e\colon x \to U$, $f\colon x \to y$, for every $\varepsilon > 0$ there exists $g\colon y \to U$ such that $\varrho(e, g \circ f) < \varepsilon$.*

We say that U is the Fraïssé limit of $\mathfrak{S}$.

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Let U be the Fraïssé limit of $\mathfrak{S}$. Then for every $\mathfrak{S}$-arrow $f\colon x \to y$, for every $\mathfrak{L}$-arrows $e_x\colon x \to U$, $e_y\colon y \to U$, for every $\varepsilon > 0$ there exists an automorphism $h\colon U \to U$ satisfying

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$$\varrho(h \circ e_x, e_y \circ f) < \varepsilon.$$

Remark

The Urysohn space is homogeneous with respect to finite sets, while the Gurariĭ space is not homogeneous with respect to finite-dimensional spaces.

Example

Let $\mathfrak{S}$ be the category whose objects are closed intervals $[0, n]$ ($n \in \mathbb{N}$) and arrows are non-expansive surjections. More precisely, $f \in \mathfrak{S}([0, n], [0, m])$ iff f is a non-expansive surjection from $[0, m]$ onto $[0, n]$.

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Fact

$\mathfrak{S}$ is a Fraïssé category, although it fails the amalgamation property.

$\mathfrak{L}$ is the category of all nonempty *chainable continua* (a continuum = a compact metrizable connected space).
The Fraïssé limit of $\mathfrak{S}$ is the *pseudo-arc*.

Bad news

Fact

The category of finite metric spaces with isometric embeddings is not countably dominated.

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The category of finite-dimensional Banach spaces with linear isometric embeddings is not countably dominated.

Proposition

A separable Banach space G is linearly isometric to the Gurariĭ space if and only if

(G) *For every finite-dimensional spaces $X \subseteq Y$, for every linear isometric embedding $e \colon X \to G$, for every $\varepsilon > 0$ there exists an ε-isometric embedding $f \colon Y \to G$ such that $\|f \restriction X - e\| < \varepsilon$.*

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Measured categories

Definition

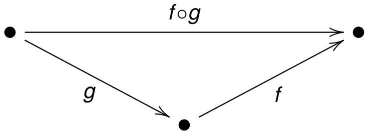
A measure on a category $\mathfrak{K}$ is a function $\mu \colon \mathfrak{K} \to [0, +\infty]$ satisfying the following conditions:

(M1) $\mu(\mathrm{id}_x) = 0$ for every object x.

(M2) $\mu(f \circ g) \leqslant \mu(f) + \mu(g)$ whenever $f \circ g$ is defined.

(M3) $\mu(g) \leqslant \mu(f \circ g) + \mu(f)$ whenever $f \circ g$ is defined.

A pair $\langle \mathfrak{K}, \mu \rangle$ will be called a measured category.



Example

Let $\mathfrak{K}$ be the category of metric spaces with non-expansive mappings. Then

$$\mu(f) = \log Lip(f^{-1})$$

defines a measure on $\mathfrak{K}$.

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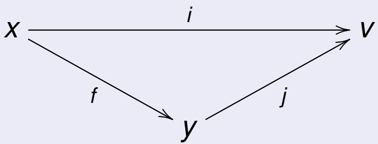
Example

Let $\mathfrak{K} = \langle X, X \times X \rangle$ be a quasi-ordered set, treated as a category such that $\mathfrak{K}(x, y) = \{\langle x, y \rangle\}$ for every $x, y \in X$. Then a measure on $\langle X, \leqslant \rangle$ is a pseudo-metric (we allow 0 for distinct points).

We assume that $\mathfrak{S}$ is a measured category enriched over metric spaces.

A new axiom

For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $f\colon x \to y$ satisfies $\mu(f) < \delta$ then there exist $i\colon x \to v$, $j\colon y \to v$ such that $\mu(i) = \mu(j) = 0$ and $\varrho(i, j \circ f) < \varepsilon$.



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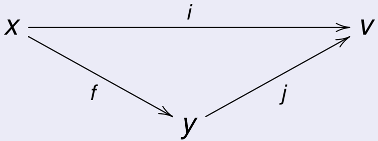

Proposition

The category of finite-dimensional Banach spaces satisfies this axiom (with $\delta = \varepsilon$).

After adapting the other assumptions and axioms, we obtain the final notion of a Fraïssé category.

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Theorem

The Urysohn space is the Fraïssé limit of the category of finite metric spaces.

Theorem

The Gurariĭ space is the Fraïssé limit of the category of finite-dimensional Banach spaces.

References, further topics

- W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Ann. Pure Appl. Logic 165 (2014) 1755–1811.
- W. Kubiś, *Banach-Mazur game played in partially ordered sets*, preprint, `http://arxiv.org/abs/1505.01094`.
- W. Kubiś, *Metric-enriched categories and approximate Fraïssé limits*, preprint, `http://arxiv.org/abs/1210.6506`.
- W. Kubiś, D. Mašulović, *Katětov functors*, preprint, `http://arxiv.org/abs/1412.1850`.