Absorption in semigroups and n-ary semigroups

Bojan Bašić

Department of Mathematics and Informatics
University of Novi Sad
Serbia

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Introduction

Definition (Barto & Kozik)

Let $\mathbf{A}$ be an algebra and $\mathbf{B} \leqslant \mathbf{A}$. We say that $\mathbf{B}$ *absorbs* $\mathbf{A}$, denoted by $\mathbf{B} \trianglelefteq \mathbf{A}$, iff there exists an idempotent term t in $\mathbf{A}$ such that for each $a \in A$ and $b_1, b_2, \ldots, b_m \in B$ we have

$$t(a, b_2, b_3, \ldots, b_m) \in B;$$
$$t(b_1, a, b_3, \ldots, b_m) \in B;$$
$$\vdots$$
$$t(b_1, b_2, b_3, \ldots, a) \in B.$$

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- A generalization of the near-unanimity (an idempotent finite algebra **A** has a near-unanimity term iff every singleton absorbs **A**).
- A very useful notion with many applications so far. For example, Bulatov's dichotomy theorem for conservative CSPs, with a deep and complicated proof (nearly 70 pages long), was reproved [Barto, 2010] using these techniques on merely 10 pages.
- Loosely speaking, the main idea of absorption is that, when $\mathbf{B} \trianglelefteq \mathbf{A}$ where **B** is a proper subalgebra of **A**, then some induction-like step can often be applied.

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- Barto & Kazda & Bulín, 2013 (announced): The absorption is decidable (a very complex algorithm).

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Theorem

Let $\mathbf{A} = (A, \cdot)$ be a semigroup, and let $\mathbf{B} \leqslant \mathbf{A}$. Then $\mathbf{B} \trianglelefteq \mathbf{A}$ if and only if $ab \in B$ and $ba \in B$ for each $a \in A$, $b \in B$, and there exists a positive integer $k > 1$ such that $a^k \approx a$ for each $a \in A$.

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- $B \ni (ab)^{d_1(r-(m-1)d_1)} t_2 t_3 \cdots t_m \approx ab$.

∎

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- Another motivation: given the very chaotic behavior of absorption in general, it is nice to have a natural class of algebras in which the absorption behaves in a very predictable (but still nontrivial) way. It might be a very useful research direction to discover whether there is a deeper reason for this nice behavior of absorption in semigroups and n-ary semigroups, and whether this reason may help to describe the behavior of absorption in other classes of algebras.

A definition of n-ary semigroup

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Definition

We say that an n-ary operation $f : A^n \longrightarrow A$ is *associative* iff

$$\begin{aligned} f(f(a_1, a_2, \ldots, a_n), a_{n+1}, \ldots, a_{2n-1}) &= f(a_1, f(a_2, \ldots, a_n, a_{n+1}), \ldots, a_{2n-1}) \\ &= \cdots \\ &= f(a_1, a_2, \ldots, f(a_n, a_{n+1}, \ldots, a_{2n-1})) \end{aligned}$$

for every $a_1, a_2, \ldots, a_{2n-1} \in A$.

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- Instead of $f(a_1, a_2, \ldots, a_n)$ we write $a_1 a_2 \cdots a_n$ etc.

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- The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are easy.

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We say that an n-ary operation f is *commutative* iff

$$f(a_1, a_2, \ldots, a_n) = f(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)})$$

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(1) $u^2vu \in B$, $2 \nmid |u|$, $2 \mid |v| \Rightarrow vu \in B$; $uvu^2 \in B \Rightarrow uv \in B$

(2) $ub \in B$, $2 \mid |u|$, $b \in B \Rightarrow bu \in B$ and vice versa

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- $a'bb = (abbab)bb = abbab^3 \approx abbab$
- $a'bba'b = abbab^3abbabb \approx abbababbabb$
- $(abb)ab(abb)^2 = abbababbabb \in B \Rightarrow abbab \in B$

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(8) $bab \in B$ for any $a \in A$, $b \in B$

(4) $aab \in B$, $baa \in B$, $aabaa \in B$ for any $a \in A$, $b \in B$

(5) whenever $t'(x, y)$ is a term such that $t'(a, b) \in B$ for all $a \in A$, $b \in B$, then $b(ab)^l \in B$, where l is the absolute value of the difference of the number of occurrences of the letter a at the odd, respectively even positions in the word $t'(a, b)$

(6) whenever $b(ab)^l \in B$ for an integer $l > 1$ and some $a \in A$, $b \in B$, then $(ab)^{l-1}a \in B$

(7) $\exists l$ such that $b(ab)^l \in B$ and $b(ab)^{l+1} \in B$ for any $a \in A$, $b \in B$

(8) $bab \in B$ for any $a \in A$, $b \in B$

(9) $ab^2 \in B$, $b^2a \in B$ for any $a \in A$, $b \in B$

■

Idempotence is not a real restriction

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Theorem

Assume that the conjecture holds for all idempotent n-ary semigroups. Then the conjecture holds in general.