On the minimal arity of near-unanimity term operations for finite algebras

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Outline

Near-unanimity operation

Definition

A near unanimity operation (NU) is an operation f satisfying

$$f(x,\ldots,x,y) = f(x,\ldots,x,y,x) = \cdots = f(y,x,\ldots,x) = x.$$

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Problem

Given a finite algebra $\mathbb{A} = (A; F)$. Decide whether there exists a near-unanimity term operation in $\mathbb{A}$.

- For any fixed n we can easily check if an algebra contains a NU term operation of arity n.
- To solve the problem we just need an upper bound on the minimal arity of a NU.

Background

Theorem (R.McKenzie, 1997)

It is undecidable for a finite algebra $\mathbb{A}$ and two elements $a, b \in A$ whether $\mathbb{A}$ has a term operation that is a near-unanimity operation on $\{a, b\}$.

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Theorem (M.Maróti, 2005)

It is decidable for a finite algebra $\mathbb{A}$ whether it has a near-unanimity term operation.

- No upper bound on the minimal arity of NU were found.

- $NU(\mathbb{A})$ denotes the minimal arity of a NU term operation in $\mathbb{A}$.
  ($NU(\mathbb{A}) = \infty$ if $\mathbb{A}$ doesn't have a NU term operation).
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- $NU_A(m) = \max\{NU(\mathbb{A}) \mid \mathrm{ar}(\mathbb{A}) \leq m, NU(\mathbb{A}) < \infty\}$,
- $NU_A^{idemp}(m)$ — the same for idempotent algebras.
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Theorem (Keith Kearnes and Ágnes Szendrei)

Suppose $\mathbb{A} = (A; f_1, \ldots, f_n)$ is a finite idempotent algebra of type $(m_1, \ldots, m_n)$. Then $NU(\mathbb{A}) = \infty$ or $NU(\mathbb{A}) \leq \sum_{i=1}^{n}(m_i - 1) + 1$.

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$NU(\mathbb{A}) \leq \sum_{i=1}^{k}(m_i - 1) + 1$.

Theorem

Suppose $m_1 \geq m_2 \geq \cdots \geq m_n$, A is a finite set,
$k = \min(n, |A|(|A| - 1)/2)$. Then there exists an idempotent algebra $\mathbb{A} = (A; f_1, \ldots, f_n)$ of type $(m_1, \ldots, m_n)$ such that
$NU(\mathbb{A}) = \sum_{i=1}^{k}(m_i - 1) + 1$.

Idempotent algebras

Suppose $m_1 \geq m_2 \geq \cdots \geq m_n$, $A = \{0, 1, \ldots, r\}$,
$n \leq |A|(|A| - 1)/2$.
Let $J_1, \ldots, J_n$ be the partition of the set $\{(a, b) \mid a < b\}$.
We define an operations f_i of arity m_i as follows

$$f_i(b, a, \ldots, a) = f_i(a, b, a, \ldots, a) = f_i(a, a, \ldots, a, b) = a$$

for all pairs $(a, b) \in J_i$,
otherwise, $f_i(x_1, \ldots, x_{m_i}) = \max(x_1, \ldots, x_{m_i})$.

Lemma

For $\mathbb{A} = (A; f_1, \ldots, f_n)$ we have $NU(\mathbb{A}) = \sum_{i=1}^{n}(m_i - 1) + 1$.

Main Results

Theorem (D.Zhuk 2013)

1. $??? \leq NU_A(m) \leq |A|^2 \cdot (|A| \cdot m)^{(3|A|)^{|A|}}$.
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All bounds hold if instead of $NU(\mathbb{A})$ we consider

- the minimal arity of an Edge term operation (- 1).
- the minimal dimension of a Cube term operation.

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Example

Let $m, k \in \mathbb{N}$, $A = \{0, 1, a_1, \ldots, a_k\}$. Put

- $h_i(x) = \begin{cases} 1, & \text{if } x \in \{1, a_i\} \\ 0, & \text{otherwise} \end{cases}$ for $i \in \{1, 2, \ldots, k\}$
- $h(\underbrace{0, 0, \ldots, 0}_{k}) = 0$, $h(\underbrace{0, 0, \ldots, 0, 1}_{i}, 0, \ldots, 0) = a_i$, otherwise h returns 1.
- $f(\underbrace{0, 0, \ldots, 0}_{m}) = 0$, $f(0, 0, \ldots, 0, a_i, 0, \ldots, 0) = 0$, otherwise f returns 1.

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Lemma

For $\mathbb{A} = (A; f, h, h_1, \ldots, h_k)$ we have $NU(\mathbb{A}) = k \cdot m$.

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Suppose $\mathbb{A} = (A; f_1, \ldots, f_n)$ is a finite idempotent algebra of type $(m_1, \ldots, m_n)$. Then $NU(\mathbb{A}) = \infty$ or $NU(\mathbb{A}) \leq \sum_{i=1}^{n}(m_i - 1) + 1$.

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- if $\mathbb{A}$ was idempotent then $NU(\mathbb{A})$ would be less than $m - 1 + k - 1 + 1 = m + k - 1$.

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Suppose $NU(\mathbb{A}) = n + 1$, then there exist $\mathbf{a}, \mathbf{b} \in A^n$ such that the tuple $\mathbf{a} \notin \mathrm{Gen}(\mathbf{a}, \mathbf{b})$.

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Suppose an algebra $\mathbb{A}$ has a cube term of dimension $n + 1$ and doesn't have a cube term of dimension n, then there exist $\mathbf{a}, \mathbf{b} \in A^n$ such that the tuple $\mathbf{a} \notin \mathrm{Gen}(\mathbf{a}, \mathbf{b})$.

For $C_1, D_1, \ldots, C_m, D_m \subseteq A$, $n_1, \ldots, n_m \in \mathbb{N}$ denote

$$\mathrm{Blob}(C_1, D_1, n_1, \ldots, C_m, D_m, n_m) = \{\mathbf{v} \in D_1^{n_1} \times D_2^{n_2} \cdots \times {D_m}^{m_n} \mid \forall j \colon C_j = \{v_{n_j - 1}, \ldots, v_{n_j}\}\}$$

The tuple $(n_1, \ldots, n_m)$ is called a type of a blob. The union of blobs of the same type is called a sponge.

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Suppose a sponge of type $(n_1, \dots, n_m)$ is an invariant of an algebra $\mathbb{A}$, $\mathrm{ar}(\mathbb{A}) < \frac{1}{|A|} \max\{n_1, n_2, \dots, n_m\}$. Then

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- $NU(\mathbb{A}) = \infty$.

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Find an exact value for $NU_A(m)$
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