Compatible Functions on Groups

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Compatible Functions on Groups

Definition

Let $\mathbf{G} = \langle G, \circ \rangle$ be a group. A function $f : G^n \to G$ is *compatible* or *congruence preserving* if

$$\forall g_1, \ldots, g_n, h_1, \ldots, h_n \in G :$$
$$f(g_1, \ldots, g_n)^{-1} \circ f(h_1, \ldots, h_n) \in \langle g_1^{-1} \circ h_1, \ldots, g_n^{-1} \circ h_n \rangle.$$

$\mathrm{Comp}_n(\mathbf{G}) = \{f : G^n \to G \mid \text{ f is compatible}\}$.

Definition

$$\pi_i^{(n)} : (x_1, \ldots, x_n) \mapsto x_i,$$
$$\bar{g} : (x_1, \ldots, x_n) \mapsto g,$$

$\mathrm{Pol}_n(\mathbf{G}) =$ subgrp. of $\mathbf{G}^{G^n}$ gen. by $\{\pi_i^{(n)} \mid i \in \{1, \ldots, n\}\} \cup \{\bar{g} \mid g \in G\}$.

Near-Rings associated with Groups

Definition

Let $\mathbf{G} = \langle G, + \rangle$ be a group, not necessarily abelian.

$\mathbf{C}_0(\mathbf{G}) = \langle C_0(\mathbf{G}), +, \circ \rangle$ is the near-ring of all unary zero-symmetric compatible functions.

$\mathbf{P}_0(\mathbf{G}) = \langle P_0(\mathbf{G}), +, \circ \rangle$ is the near-ring of all unary zero-symmetric polynomial functions.

$\mathbf{I}(\mathbf{G}) = \mathbf{P}_0(\mathbf{G})$ is the near-ring generated by all inner-automorphisms.

$\mathbf{G}$ is 1-*affine complete* if $\mathbf{C}_0(\mathbf{G}) = \mathbf{I}(\mathbf{G})$.

Inner-Automorphism Near-Ring

Theorem (A. John Chandy, 1971)

G *a group. The inner-automorphism near-ring* $\mathbf{I}(\mathbf{G})$ *is a ring iff* $\mathbf{G} \models [x,[x,y]] = 1$.

These groups are called 2-Engel groups. Every 2-nilpotent group is a 2-Engel group.

Corollary (A. John Chandy, 1971)

If the group **G** *is* 2*-nilpotent then the inner-automorphism ring is commutative.*

```
gap> G:=AbelianGroup([8,2]);; #C8 x C2
gap> I:=InnerAutomorphismNearRing(G);;
gap> IsDistributiveNearRing(I);
true
gap> IsCommutative(G);
true
```

Zero-Symmetric Compatible Functions

Problem

Given a group $\mathbf{G}$*. When is* $\mathbf{C}_0(\mathbf{G})$ *a ring?*

For example $\mathbf{C}_0(\mathbf{G})$ is a ring for:

1. 1-affine complete 2-Engel groups and
2. 1-affine complete abelian groups.

```
gap> G:=SmallGroup(16,3);; #(C4 x C2) : C2
gap> Co:=ZeroSymmetricCompatibleFunctionNearRing(G);;
gap> I:=InnerAutomorphismNearRing(G);;
gap> Size(Co)=Size(I);
false
gap> IsDistributiveNearRing(Co);
true
```

Properties of Lattices

We will now investigate congruence lattices.

Definition

$(\delta, \varepsilon) \in \mathbf{L}^2$ is a *splitting pair* of the lattice $\mathbf{L}$ if $\delta < 1, \varepsilon > 0$ and for all $\alpha \in \mathbf{L}$, we have $\alpha \leq \delta$ or $\alpha \geq \varepsilon$.

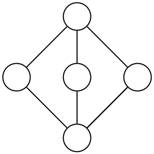
The lattice $\mathbf{L}$ *splits* if it has a splitting pair.

Definition

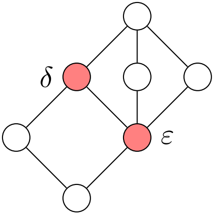
$\beta \in \mathbf{L}$ is a *cutting element* of the lattice $\mathbf{L}$ if $\beta < 1, \beta > 0$ and for all $\alpha \in \mathbf{L}$ we have $\alpha \leq \beta$ or $\alpha \geq \beta$.

The lattice $\mathbf{L}$ *cuts* if it has a cutting element.

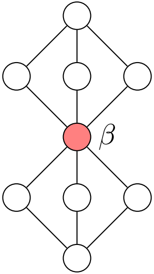
Properties of Lattices



(a) $\mathbb{M}_3$ does not split



(b) splits



(c) cuts

Necessary Conditions for $C_0(\mathbf{G})$ to be a ring

Theorem (FS)

If $\mathbf{G}$ is abelian and $C_0(\mathbf{G})$ is a ring, then $\mathbf{G}$ is 1-affine complete.

Since every abelian group is also a 2-Engel group, we immediately get:

Corollary

If $\mathbf{G}$ is abelian then $C_0(\mathbf{G})$ is a ring iff $\mathbf{G}$ is 1-affine complete.

The 1-affine completeness of abelian groups has been characterized by Lausch and Nöbauer in 1973.

Theorem (FS)

If $Con(\mathbf{G})$ cuts then $C_0(\mathbf{G})$ is not a ring.

Necessary Conditions for $C_0(\mathbf{G})$ to be a ring

Theorem (FS)

Let $\mathbf{G}$ *be a group such that* $C_0(\mathbf{G})$ *is a ring. Let* $(\delta, \varepsilon) \in \mathrm{Con}\,(\mathbf{G})^2$ *be a splitting pair of* $\mathrm{Con}\,(\mathbf{G})$*. Then we have that* $\mathbf{G}/\delta \cong \mathbb{Z}_2^n$ *and* $\varepsilon \cong \mathbb{Z}_2^m$

Corollary

If $\mathrm{Con}\,(\mathbf{G})$ *splits and* $2 \nmid |\mathbf{G}|$ *then* $C_0(\mathbf{G})$ *is not a ring.*

Especially for every odd prime p and for every p-group such that $\mathrm{Con}\,(\mathbf{G})$ splits, $C_0(\mathbf{G})$ is not a ring.

Affine Complete Groups

Definition

G is *n-affine complete* if $\mathrm{Comp}_n(\mathbf{G}) = \mathrm{Pol}_n(\mathbf{G})$ and **G** is *affine complete* if **G** is *n*-affine complete for all $n \in \mathbb{N}$.

Theory to determine affine completeness exists for the following classes of groups:

1. abelian groups [Lausch, Nöbauer, 1973]
2. groups with APMI [Aichinger, Mudrinski, 2009]
3. groups with splitting congruce lattice [Aichinger 2002]

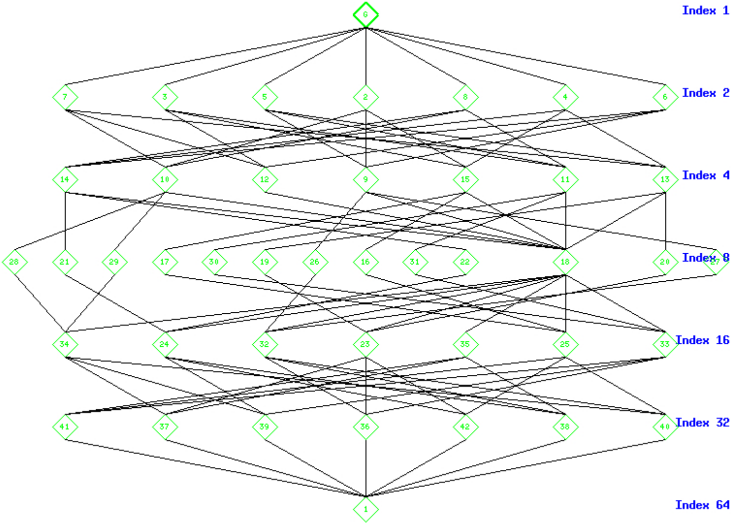
Affine Complete Groups

The theory covers all groups of order up to 100 except for the following

1. $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_4) \rtimes_{\alpha_1} \mathbb{Z}_2$ and
2. $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes_{\alpha_2} Q_8$.

Both groups have a non-splitting normal subgroup lattice and are of nilpotency class 2.

Normal Subgroup Lattice



Affine Complete Groups

Theorem (E. Aichinger, J. Ecker)

Let **H** *be a group of nilpotency class* k. *If* **H** *is* $(k+1)$*-affine complete, then* **H** *is affine complete.*

Since **G** is 2-nilpotent it suffices to check if **G** is 3-affine complete.

1-affine completeness:

```
gap> G:=SmallGroup(64,73);;
gap> StructureDescription(G);
"(C2 x C2 x D8) : C2"
gap> Comp:=CompatibleFunctionNearRing(G);;
gap> Pol:=PolynomialNearRing(G);;
gap> Size(Pol);Size(Comp);
2048
2048
```

Affine Complete Groups

Let $\mathbf{V}$ be a finite expanded group, $n \in \mathbb{N}_0$ and let

$$a_n(\mathbf{V}) := \log_2(|\{p \in \mathrm{Pol}_n(\mathbf{V}) | p \text{ is absorbing}\}|)$$
$$p_n(\mathbf{V}) := \log_2(|\mathrm{Pol}_n(\mathbf{V})|).$$

Theorem (Higman, Berman, Blok)

Let $\mathbf{V}$ be a finite expanded group. Then for each $n \in \mathbb{N}_0$, we have

$$p_n(\mathbf{V}) = \sum_{i=0}^{n} a_i(\mathbf{V}) \binom{n}{i}.$$

Affine Complete Groups

Let $\bar{\mathbf{G}} := \langle G, Comp(\mathbf{G})\rangle$.

$$p_n(\bar{\mathbf{G}}) = \sum_{i=0}^{n} a_i(\bar{\mathbf{G}}) \binom{n}{i}.$$

First one can see that $a_0(\bar{\mathbf{G}}) = 6$ and $p_1(\bar{\mathbf{G}}) = 11$. That gives us that $a_1(\bar{\mathbf{G}}) = 5$.
Next we show that $\bar{\mathbf{G}}$ has only 2 absorbing polynomial functions, thus $a_2(\bar{\mathbf{G}}) = 1$. Hence $p_2(\bar{\mathbf{G}}) = p_2(\mathbf{G}) = 17$, meaning that $\mathbf{G}$ is 2-affine complete.

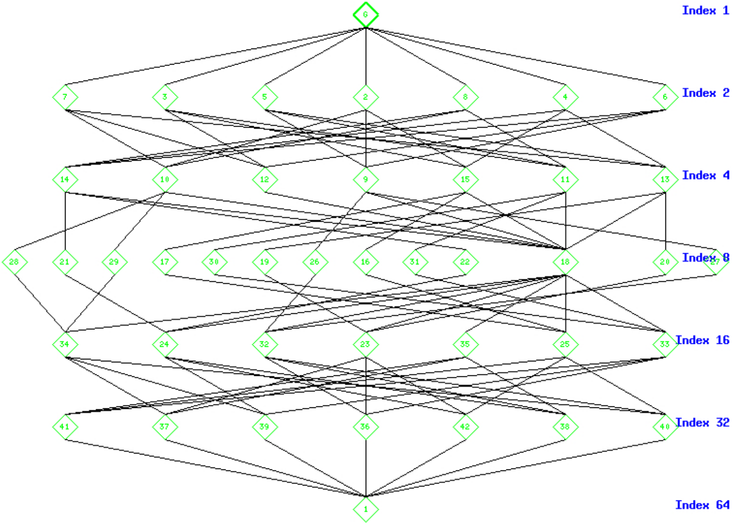
Next we prove $a_3(\bar{\mathbf{G}}) = 0$. To this end we show that Con $(\bar{\mathbf{G}})$ does not admit a nontrivial ternary commutator operation.

The ternary commutator is defined as $[\cdot,\cdot,\cdot] : Con(\bar{\mathbf{G}})^3 \to Con(\bar{\mathbf{G}})$.
Proving $a_3(\bar{\mathbf{G}}) = 0$ we need: $\forall \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2 \in Con(G)$:

1. (HC1) $[\alpha_1, \alpha_2, \alpha_3] \leq \alpha_1 \wedge \alpha_2 \wedge \alpha_3$
2. (HC2) if $\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2, \alpha_3 \leq \beta_3$ then $[\alpha_1, \alpha_2, \alpha_3] \leq [\beta_1, \beta_2, \beta_3]$
3. (HC3) $[\alpha_1, \alpha_2, \alpha_3] \leq [\alpha_1, \alpha_2]$
4. (HC4) $[\alpha_1, \alpha_2, \alpha_3] = [\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}]$ for $\sigma \in S_3$
5. (HC7) $[\gamma_1 \vee \gamma_2, \alpha_2, \alpha_3] = [\gamma_1, \alpha_2, \alpha_3] \vee [\gamma_2, \alpha_2, \alpha_3]$.

Since the ternary commutator of $\bar{\mathbf{G}}$ is trivial we know that $a_3(\bar{\mathbf{G}}) = 0$.
Thus $\mathbf{G}$ has to be 3-affine complete and therefore affine complete.

Normal Subgroup Lattice



Affine Complete Groups

$\mathbf{G} := (\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_4) \rtimes_{\alpha_1} \mathbb{Z}_2$:

G is defined in the following way. Take $N := \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_4$ with $N = \langle g_1, g_2, a, b \rangle$ where $\mathrm{ord}(a) = 4, \mathrm{ord}(b) = 2$.
Let $\beta \in \mathrm{Aut}(N)$ be defined as follows:

$$\begin{aligned} \beta : N &\to N \\ g_1 &\mapsto g_1 \\ g_2 &\mapsto g_2 \\ a &\mapsto g_1 * a \\ b &\mapsto g_2 * b. \end{aligned}$$

Then α_1 is defined in the following way:

$$\begin{aligned} \alpha_1 : \mathbb{Z}_2 &\to \mathrm{Aut}(N) \\ 1_{\mathbb{Z}_2} &\mapsto \beta. \end{aligned}$$