Abelian quandles

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Quasi-affine algebras

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$\mathbf{A}$ satisfies

$$t(x, u_1, \ldots, u_k) = t(x, v_1, \ldots, v_k) \quad \Rightarrow \quad t(y, u_1, \ldots, u_k) = t(y, v_1, \ldots, v_k)$$

for every term t.

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Remark

Quasi-affine algebras $\mathbf{A}$ *are abelian.*

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- $\mathbf{A}$ is **entropic** if all basic operations are central

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- (K.Kearnes) **A** is an abelian, simple, idempotent algebra ⇒ **A** is quasi-affine.

Abelian + idempotent $\Leftrightarrow$ quasi-affine

Open problem

Is it true that every idempotent abelian algebra is quasi-affine?

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- (D.Stanovský) Abelian differential modes are quasi-affine.

(left, n-ary) **differential mode** (A,f):

- $f(x,\ldots,x)=x$
- $f(f(x,y_2,\ldots,y_n),z_2,\ldots,z_n)=f(f(x,z_2,\ldots,z_n),y_2,\ldots,y_n)$
- $f(x,f(y_{21},\ldots,y_{2n}),\ldots,f(y_{n1},\ldots,y_{nn}))=f(x,y_{21},\ldots,y_{n1})$

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Conjecture

All abelian modes are quasi-affine.

Quandles

Definition

A binary algebra $(Q, \cdot)$ is called a **quandle** if it is:

- **left distributive**: $x(yz) = (xy)(xz)$ for every $x, y, z \in Q$
- **idempotent**: $xx = x$ for each $x \in Q$
- a **left quasigroup**: the equation $xu = y$ has a unique solution $u \in Q$ for every $x, y \in Q$

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- $(A,+)$ - an abelian group, $f \in Aut(A,+)$
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Remark

A quandle is quasi-affine if it embeds into an affine quandle.

Affine quandles

The **displacement group** - the subgroup of $Aut(Q, \cdot)$:

$$\mathrm{Dis}(Q) = \langle L_a L_b^{-1} \mid a, b \in Q \rangle.$$

For $a \in Q$, $L_a : Q \to Q$, $x \mapsto ax$

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- Affine quandles are medial.
- A quandle $(Q,\cdot)$ is medial if and only if $\mathrm{Dis}(Q)$ is commutative.

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Theorem (JPSZ)

A quandle Q is abelian iff

- $\mathrm{Dis}(Q)$ is commutative
- the only mapping from $\mathrm{Dis}(Q)$ with a fixed point is the identity mapping

The structure of medial quandles

affine mesh = triple $((A_i)_{i\in I},\ (\varphi_{i,j})_{i,j\in I},\ (c_{i,j})_{i,j\in I})$ indexed by I where

- A_i are abelian groups
- $\varphi_{i,j} : A_i \to A_j$ homomorphisms
- $c_{i,j} \in A_j$ constants

such that for every $i,j,j',k \in I$

- $\mathrm{id} - \varphi_{i,i}$ is an automorphism of A_i
- $c_{i,i} = 0$
- $\varphi_{j,k}\varphi_{i,j} = \varphi_{j',k}\varphi_{i,j'}$ (they commute naturally)
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sum of an affine mesh = disjoint union of A_i, for $a \in A_i$, $b \in A_j$

$$a * b = c_{i,j} + \varphi_{i,j}(a) + (\mathrm{id} - \varphi_{j,j})(b)$$

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Theorem (JPSZ)

An algebra is a medial quandle if and only if it is the sum of an affine mesh.

Abelian quandles as the sum of an affine mesh

Theorem (JPSZ)

Each abelian quandel Q is the sum of an affine mesh $\mathcal{A} = ((A, A, ...); \varphi; (c_{i,j})_{i,j \in I})$ over a non-empty set I and $A \simeq \mathrm{Dis}(Q)$.

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  Q is an affine quandle.

- **Case 2.** $\varphi \notin \mathrm{Aut}(A)$. None of the orbits is a quasigroup.

Not all abelian quandles are affine quandles.

Example

Q - the sum of the affine mesh: $((Z_3, Z_3); \varphi = 0; \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$

$*$	0	1	2	3	4	5
0	0	1	2	4	5	3
1	0	1	2	4	5	3
2	0	1	2	4	5	3
3	1	2	0	3	4	5
4	1	2	0	3	4	5
5	1	2	0	3	4	5

Q is not an affine quandle.

Main theorem

Theorem (JPSZ)

Each abelian quandel is quasi-affine.

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Proof.

Idea: To verify the axioms of quasi-affine algebras presented by M.Stronkowski and D.Stanovský in *Embedding general algebras into modules*, Proc. Amer. Math. Soc. 138/8 (2010). □

Case 2. $\varphi \notin \mathrm{Aut}(A)$

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$S \subseteq Q$ - a transwersal of the partition by the relation $\ker R_e$

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A non 2-reductive abelian quandle is affine iff it satisfies the condition (1).

Thank you for your attention!