Local monotonicities and lattice derivatives of Boolean and pseudo-Boolean functions

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Partial derivatives

- Boolean function: \( f : \{0, 1\}^n \rightarrow \{0, 1\} \)
Partial derivatives

- **Boolean function**: $f : \{0, 1\}^n \rightarrow \{0, 1\}$
- **pseudo-Boolean function**: $f : \{0, 1\}^n \rightarrow \mathbb{R}$
Partial derivatives

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- **pseudo-Boolean function**: $f : \{0, 1\}^n \rightarrow \mathbb{R}$
- The partial derivative of $f : \{0, 1\}^n \rightarrow \mathbb{R}$ w.r.t. $x_k$ is the function $\Delta_k f : \{0, 1\}^n \rightarrow \mathbb{R}$ defined by

  $$\Delta_k f(x) = f(x^1_k) - f(x^0_k) = f(x_1, \ldots, 1, \ldots, x_n) - f(x_1, \ldots, 0, \ldots, x_n).$$

  Observe that $\Delta_k f$ does not depend on $x_k$. 
Partial derivatives

- Boolean function: \( f : \{0, 1\}^n \rightarrow \{0, 1\} \)
- pseudo-Boolean function: \( f : \{0, 1\}^n \rightarrow \mathbb{R} \)
- The partial derivative of \( f : \{0, 1\}^n \rightarrow \mathbb{R} \) w.r.t. \( x_k \) is the function \( \Delta_k f : \{0, 1\}^n \rightarrow \mathbb{R} \) defined by

\[
\Delta_k f(x) = f(x_k^1) - f(x_k^0) = f(x_1, \ldots, 1, \ldots, x_n) - f(x_1, \ldots, 0, \ldots, x_n).
\]

Observe that \( \Delta_k f \) does not depend on \( x_k \).

Example

The partial derivatives of the Boolean sum
\( f(x_1, x_2) = x_1 \oplus x_2 = x_1 + x_2 - 2x_1x_2 \) are

\[
\Delta_1 f(x_1, x_2) = f(1, x_2) - f(0, x_2) = 1 - 2x_2, \\
\Delta_2 f(x_1, x_2) = f(x_1, 1) - f(x_1, 0) = 1 - 2x_1.
\]
Monotonicity

- $f$ is isotone (positive, order-preserving, nondecreasing) in $x_k$ if

$$\Delta_k f(x) \geq 0 \text{ for all } x \in \{0, 1\}^n.$$
Monotonicity

- \( f \) is isotone (positive, order-preserving, nondecreasing) in \( x_k \) if
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- \( f \) is antitone (negative, order-reversing, nonincreasing) in \( x_k \) if
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  \Delta_k f(x) \leq 0 \text{ for all } x \in \{0, 1\}^n.
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f is isotone (positive, order-preserving, nondecreasing) in $x_k$ if

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f is monotone in $x_k$ if it is either isotone or antitone in $x_k$, i.e., if $\Delta_k f(x)$ does not change sign.
Monotonicity

- $f$ is **isotone** (positive, order-preserving, nondecreasing) in $x_k$ if
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- $f$ is **monotone (isotone, antitone)** if it is monotone (isotone, antitone) in all of its variables.
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- **f** is **monotone** (**isotone, antitone**) if
  it is monotone (**isotone, antitone**) in all of its variables.

- All unary functions are monotone.
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- **f** is monotone (isotone, antitone) if it is monotone (isotone, antitone) in all of its variables.

- All unary functions are monotone.

- The only non-monotone binary Boolean functions are $x_1 \oplus x_2$ and $x_1 \oplus x_2 \oplus 1.$
We say that $f: \{0, 1\}^n \rightarrow \mathbb{R}$ is \textit{p}-locally monotone if, for every $k \in [n]$ and every $x, y \in \{0, 1\}^n$, we have

$$\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(x) \Delta_k f(y) \geq 0.$$
Local monotonicities

Definition

We say that $f : \{0, 1\}^n \rightarrow \mathbb{R}$ is $p$-locally monotone if, for every $k \in [n]$ and every $x, y \in \{0, 1\}^n$, we have

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$p$-local monotonicity implies $(p - 1)$-local monotonicity.
Local monotonicities

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$$\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \implies \Delta_k f(x) \Delta_k f(y) \geq 0.$$  

- $p$-local monotonicity implies $(p - 1)$-local monotonicity.
- An $n$-ary function is $n$-locally monotone iff it is monotone.
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\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(x) \Delta_k f(y) \geq 0.
\]

- \( p \)-local monotonicity implies \((p - 1)\)-local monotonicity.
- An \( n \)-ary function is \( n \)-locally monotone iff it is monotone.
- Every function is \( 1 \)-locally monotone.
Local monotonicities

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\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(x) \Delta_k f(y) \geq 0.
\]

- \( p \)-local monotonicity implies \((p - 1)\)-local monotonicity.
- An \( n \)-ary function is \( n \)-locally monotone iff it is monotone.
- Every function is 1-locally monotone.

**Theorem**

A Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is 2-locally monotone iff

\[
|\Delta_k f(x) - \Delta_k f(y)| \leq \sum_{i \in [n] \setminus \{k\}} |x_i - y_i|.
\]
We define the \textbf{partial lattice derivatives} of $f : \{0, 1\}^n \to \mathbb{R}$ w.r.t. $x_k$ by

\begin{align*}
\wedge_k f &: \{0, 1\}^n \to \mathbb{R}, \quad \wedge_k f(x) = f(x_0^k) \wedge f(x_1^k) = \min(f(x_0^k), f(x_1^k)), \\
\vee_k f &: \{0, 1\}^n \to \mathbb{R}, \quad \vee_k f(x) = f(x_0^k) \vee f(x_1^k) = \max(f(x_0^k), f(x_1^k)).
\end{align*}
Lattice derivatives

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\]

Example

The lattice derivatives of the Boolean sum \( f(x_1, x_2) = x_1 \oplus x_2 \) are

\[
\wedge_1 f(x_1, x_2) = f(1, x_2) \wedge f(0, x_2) = (1 \oplus x_2) \wedge x_2 = 0,
\]

\[
\vee_1 f(x_1, x_2) = f(1, x_2) \vee f(0, x_2) = (1 \oplus x_2) \vee x_2 = 1.
\]

The second-order lattice derivatives are

\[
\vee_2 \wedge_1 f(x_1, x_2) = \vee_2 0 = 0,
\]

\[
\wedge_1 \vee_2 f(x_1, x_2) = \wedge_1 1 = 1.
\]
A Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is 2-locally monotone iff

$$\bigvee_k \bigwedge_j f = \bigwedge_j \bigvee_k f \text{ for all } j \neq k.$$
Theorem

A Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is 2-locally monotone iff

$$\bigvee_k \bigwedge_j f = \bigwedge_j \bigvee_k f \text{ for all } j \neq k.$$ 

Definition

We say that $f : \{0, 1\}^n \rightarrow \mathbb{R}$ has $p$-permutable lattice derivatives, if

$$O_{k_1} \cdots O_{k_p} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f$$

holds for every $p$-element set $\{k_1, \ldots, k_p\} \subseteq \{1, \ldots, n\}$, for all operators $O_{k_i} \in \{\bigwedge_{k_i}, \bigvee_{k_i}\}$ and for every permutation $\pi \in S_p$. 
Permutable lattice derivatives

**Theorem**

A Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is 2-locally monotone iff

\[ \bigvee_k \bigwedge_j f = \bigwedge_j \bigvee_k f \text{ for all } j \neq k. \]

**Definition**

We say that \( f : \{0, 1\}^n \rightarrow \mathbb{R} \) has \( p \)-permutable lattice derivatives, if

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holds for every \( p \)-element set \( \{k_1, \ldots, k_p\} \subseteq \{1, \ldots, n\} \), for all operators \( O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\} \) and for every permutation \( \pi \in S_p \).

**Theorem**

If a function has \( p \)-permutable lattice derivatives, then it has \( (p - 1) \)-permutable lattice derivatives.
Theorem

If a function is $p$-locally monotone, then it has $p$-permutable lattice derivatives.
Theorem

If a function is $p$-locally monotone, then it has $p$-permutable lattice derivatives.

Example

Let $f : \{0, 1\}^n \to \{0, 1\}$ be the function that takes the value 0 on all tuples of the form

$$(1, \ldots, 1, 0, \ldots, 0)$$

with $0 \leq m \leq n$,

and takes the value 1 everywhere else. Then $f$ has $n$-permutable lattice derivatives, but it is only 2-locally monotone.
Theorem

If a function is $p$-locally monotone, then it has $p$-permutable lattice derivatives.

Example

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be the function that takes the value 0 on all tuples of the form

\[(1, \ldots, 1, 0, \ldots, 0) \text{ with } 0 \leq m \leq n,\]

and takes the value 1 everywhere else. Then $f$ has $n$-permutable lattice derivatives, but it is only 2-locally monotone.

Theorem

For symmetric functions, $p$-local monotonicity is equivalent to $p$-permutability of lattice derivatives.
A section of a function $f$ is any function $g$ that can be obtained from $f$ by substituting constants to some of the variables of $f$.

For example, if $f: \{0, 1\}^3 \to \mathbb{R}$, then $g: \{0, 1\}^2 \to \mathbb{R}$, $g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of $f$. 
A section of a function $f$ is any function $g$ that can be obtained from $f$ by substituting constants to some of the variables of $f$.

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Sections

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For example, if $f : \{0, 1\}^3 \to \mathbb{R}$, then $g : \{0, 1\}^2 \to \mathbb{R}$, $g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of $f$. 

![Diagram](image)
Theorem

If a function is nice, then all of its sections are also nice, where “nice” stands for any of the previously discussed properties.
Forbidden sections

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Corollary

A function is nice iff none of the minimal ugly functions appear among its sections.
Forbidden sections

Theorem

A Boolean function is isotone iff \( x_1 \oplus 1 \) does not appear among its sections.
Forbidden sections

**Theorem**

A Boolean function is isotone iff $x_1 \oplus 1$ does not appear among its sections.

**Theorem**

A Boolean function is 2-locally monotone iff neither $x_1 \oplus x_2$ nor $x_1 \oplus x_2 \oplus 1$ appears among its sections.
Forbidden sections

**Theorem**

A Boolean function is isotone iff $x_1 \oplus 1$ does not appear among its sections.

**Theorem**

A Boolean function is 2-locally monotone iff neither $x_1 \oplus x_2$ nor $x_1 \oplus x_2 \oplus 1$ appears among its sections.

**Conjecture**

A Boolean function has permutable lattice derivatives iff none of the following functions appear among its sections:

![Diagram of lattice structures]
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http://www.math.u-szeged.hu/algebra2012

Dedicated to the 80th birthday of Béla Csákány