A relational localisation theory for topological algebras

Friedrich Martin Schneider

Technische Universität Dresden

Novi Sad, March 17, 2012
What will this talk be about?

I will sketch a general Galois theory for continuous operations and closed relations on a topological space and characterise the corresponding system of Galois closures. I will introduce a relational localisation theory for topological algebras, identify suitable subsets, describe the restriction process and explain how to reconstruct an algebra from its decomposition. I will explore the developed concepts for modules of compact rings.
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► introduce a relational **localisation theory** for topological algebras, identify suitable subsets, describe the restriction process and explain how to reconstruct an algebra from its decomposition.

► explore the developed concepts for **modules of compact rings**.
The Galois connection $c\text{Pol}-c\text{Inv}$

Let $X = (A, T)$ be a topological space, $m, n \in \mathbb{N}$.
The Galois connection \( \text{cPol}-\text{cInv} \)

Let \( X = (A, T) \) be a topological space, \( m, n \in \mathbb{N} \).

\[
O_A^{(n)} := A^A^n, \quad R_A^{(m)} := \mathcal{P}(A^m), \\
O_A := \bigcup_{n \in \mathbb{N}} O_A^{(n)}, \quad R_A := \bigcup_{m \in \mathbb{N}} R_A^{(m)}, \\
cO_{X}^{(n)} := C(X^n; X), \quad cR_{X}^{(m)} := \{ \varrho \subseteq A^m \mid \varrho \text{ closed in } X^m \}, \\
cO_X := \bigcup_{n \in \mathbb{N}} cO_{X}^{(n)}, \quad cR_X := \bigcup_{m \in \mathbb{N}} cR_{X}^{(m)}.
\]
The Galois connection $\text{cPol-clInv}$

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- $O_A := \bigcup_{n \in \mathbb{N}} O_A^{(n)}$,
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- $cO_X^{(n)} := C(X^n; X)$,
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- $cO_X := \bigcup_{n \in \mathbb{N}} cO_X^{(n)}$,
- $cR_X := \bigcup_{m \in \mathbb{N}} cR_X^{(m)}$.

For $f \in O_A^{(n)}$ and $\varrho \in R_A^{(m)}$,
The Galois connection cPol-clnv

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$$cR_X^{(m)} := \{ \varrho \subseteq A^m \mid \varrho \text{ closed in } X^m \},$$
$$cR_X := \bigcup_{m \in \mathbb{N}} cR_X^{(m)}.$$

For $f \in O_A^{(n)}$ and $\varrho \in R_A^{(m)}$,

$$f \triangleright \varrho \iff \forall r_0, \ldots, r_{n-1} \in \varrho : f \circ \langle r_0, \ldots, r_{n-1} \rangle \in \varrho$$
$$\iff \varrho \in \text{Sub}(\langle A; f \rangle^m)$$
$$\iff f \in \text{Hom}(\langle A; \varrho \rangle^n; \langle A; \varrho \rangle).$$
The Galois connection $cPol$-$cInv$ (cont’d.)

For $F \subseteq cO_X$, $cInv \langle A, T, F \rangle := cInv_X F := \{ \varrho \in cR_X \mid \forall f \in F: f \varrho \}$.

for $Q \subseteq cR_X$, $cPol \langle A, T, Q \rangle := cPol_X Q := \{ f \in cO_X \mid \forall \varrho \in Q: f \varrho \}$.

How can we describe the closure system induced by this Galois connection?
The Galois connection \( cPol-cInv \) (cont’d.)

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For \( F \subseteq cO_X \),

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\]

How can we describe the closure system induced by this Galois connection?
Clones of operations

A set $F \subseteq O_A$ is called clone of operations on $A$ if

1. $F$ contains all projections,
2. for $m, n \in \mathbb{N}$, $f \in F(n)$, $f_0, \ldots, f_{n-1} \in F(m)$, we also have $f \circ \langle f_0, \ldots, f_{n-1} \rangle \in F$.

For any set $F \subseteq O_A$, the smallest clone on $A$ containing $F$ is denoted by $\text{Clo}(F)$.

Obviously, $cO_X$ is a clone of operations on $A$. 
Clones of operations

Reminder
A set $F \subseteq O_A$ is called clone of operations on $A$ if

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Clones of closed relations

A set $Q \subseteq cR^X$ is called clone of closed relations on $X$ if $Q$ is closed w.r.t. general superposition of closed relations, that is:

Whenever $I$ is a set, $Y = (B, S)$ a topological space, $m, m_i \in \mathbb{N}$, $\phi: m \rightarrow B$, $\phi_i: m_i \rightarrow B$ and $\varrho_i \in Q(m_i)$ for $i \in I$, then

$\bigwedge_{i \in I} \phi, Y, X(\phi_i) \in Q$,

where $\bigwedge_{i \in I} \phi, Y, X(\phi_i) := \{ r \circ \phi | r \in C(Y; X), \forall i \in I: r \circ \phi_i \in \varrho_i \}$.

For any set $Q \subseteq cR^X$, the smallest clone of closed relations on $X$ containing $Q$ is denoted by $\text{CLO}(Q)$. 
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A set $Q \subseteq cR_\mathcal{X}$ is called clone of closed relations on $\mathcal{X}$ if $Q$ is closed w.r.t. general superposition of closed relations, that is:
Whenever $I$ is a set, $\mathcal{Y} = (B, S)$ a topological space, $m, m_i \in \mathbb{N}$, $\varphi : m \to B$, $\varphi_i : m_i \to B$ and $\varrho_i \in Q^{(m_i)}$ for $i \in I$, then

\[
\bigwedge_{(\varphi_i)_{i \in I}} (\varrho_i)_{i \in I} \in Q,
\]

where

\[
\bigwedge_{(\varphi_i)_{i \in I}} (\varrho_i)_{i \in I} := \{ r \circ \varphi | r \in C(\mathcal{Y}; \mathcal{X}), \forall i \in I : r \circ \varphi_i \in \varrho_i \}.
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$$\bigwedge_{(\varphi_i)_{i \in I}}^{(\varrho_i)_{i \in I}} X^m \in Q,$$

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$$\bigwedge_{(\varphi_i)_{i \in I}}^{(\varrho_i)_{i \in I}} := \{ r \circ \varphi \mid r \in C(Y; X), \forall i \in I : r \circ \varphi_i \in \varrho_i \}.$$

For any set $Q \subseteq cR_X$, the smallest clone of closed relations on $X$ containing $Q$ is denoted by $\text{CLO}(Q)$.
Local closure operators

Definition

For $F \subseteq cO_{\mathcal{X}}$, $Q \subseteq cR_{\mathcal{X}}$ and $s \in \mathbb{N}$:

$s$-Loc$_F := \{ f \in cO_{\mathcal{N}(n)} \ | \ n \in \mathbb{N}, \forall a \in (A^n)_s, U \in T_s : [ f(a_0) \in U_0, \ldots, f(a_{s-1}) \in U_{s-1}] \Rightarrow \exists g \in F : g(a_0) \in U_0, \ldots, g(a_{s-1}) \in U_{s-1} \}$,

$s$-Loc$_Q := \{ \varrho \in cO_{\mathcal{X}} | \forall \sigma \subseteq \varrho, |\sigma| \leq s : \exists \varrho' \in Q : \sigma \subseteq \varrho' \subseteq \varrho \}$,

Loc$_F := \bigcap_{s \in \mathbb{N}} s$-Loc$_F$, LOC$_Q := \bigcap_{s \in \mathbb{N}} s$-Loc$_Q$. 
Local closure operators

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For $F \subseteq cO_X$, $Q \subseteq cR_X$ and $s \in \mathbb{N}$:

$$s\text{-Loc} F := \{ f \in cO_X(n) \mid n \in \mathbb{N}, \forall a \in \{ A_n \}^s, U \in T^s : f(a_0) \in U_0, \ldots, f(a_{s-1}) \in U_{s-1} \}$$

$$s\text{-LOC} Q := \{ \varrho \in cO_X \mid \forall \sigma \subseteq \varrho, |\varrho| \leq s : \exists \varrho' \in Q : \sigma \subseteq \varrho' \subseteq \varrho \}$$

$$\text{Loc} F := \bigcap_{s \in \mathbb{N}} s\text{-Loc} F$$

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Local closure operators

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\begin{align*}
\text{s-Loc } F & := \{ f \in cO_X^n \mid n \in \mathbb{N}, \forall a \in (A^n)^s, U \in T^s : \\
& \quad [f(a_0) \in U_0, \ldots, f(a_{s-1}) \in U_{s-1}] \Rightarrow \\
& \quad [\exists g \in F : g(a_0) \in U_0, \ldots, g(a_{s-1}) \in U_{s-1}] \}, \\
\text{s-LOC } Q & := \{ \varrho \in cO_X \mid \forall \sigma \subseteq \varrho, |\sigma| \leq s : \exists \varrho' \in Q : \sigma \subseteq \varrho' \subseteq \varrho \}, \\
\text{Loc } F & := \bigcap_{s \in \mathbb{N}} \text{s-Loc } F, \\
\text{LOC } Q & := \bigcap_{s \in \mathbb{N}} \text{s-LOC } Q.
\end{align*}
$$
Characterising the Galois closures

Theorem
Let $F \subseteq \mathbb{cO}_X$. Then:
(a) $s$-Loc Clo($F$) = $cPol$ $cInv$ ($s$)$F$ for $s \in \mathbb{N}$.
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Let $F \subseteq cO_X$. Then:

(a) $s\text{-Loc } \text{Clo}(F) = cPol cInv^{(s)} F$ for $s \in \mathbb{N}$.

(b) $\text{Loc } \text{Clo}(F) = cInv cPol F$. 

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A localisation theory consists of three main ingredients.

1. Localisation: Restricting the structure to suitable subsets.
2. Classification: Calculating locally.
3. Globalisation: Combining local results into global results.

What are the suitable subsets for this kind of localisation theory?
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Topologising RST

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Finding suitable subsets

Let $A = \langle A; T; F \rangle$ be a topological algebra. For $U \subseteq A$, \( E_A(U) := \{ e \mid e \in \text{Loc Clo}(1)(F), \text{im} e \subseteq U \} \).

Lemma

Let $U \subseteq A$. The following are equivalent:

(a) $\cdot \upharpoonright U : \text{cInv} A \to \text{cR}(U, T_U)$, $\varrho \mapsto \varrho \upharpoonright U$ is a homomorphism between clones of closed relations.

(b) $\text{id}_U \in \text{Loc} \{ e \mid U \subseteq e \mid e \in E_A(U) \}$.

Additionally, if (a) holds, then $[Q] \upharpoonright U := \{ \varrho \upharpoonright U \mid \varrho \in Q \}$ is locally closed for every locally closed clone of closed relations $Q \subseteq \text{cInv} A$. 
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Additionally, if (a) holds, then

$$[Q] \upharpoonright_U := \{ \varrho \upharpoonright_U \mid \varrho \in Q \}$$

is locally closed

for every locally closed clone of closed relations $Q \subseteq \text{clnv } A$. 
Restricting algebras to neighbourhoods

\[ A \not\equiv A = \langle A, T, cInv A \rangle \]
\[ cInv \langle A, T, Loc \ Clo (A) \rangle \]
\[ cPol \equiv \top \ cInv \]
\[ \overset{\equiv_{\text{top}}}{A} \rightarrow cInv \]
\[ \langle A, T, Loc \ Clo (A) \rangle \]
\[ \overset{cPol}{\leftarrow} \]
\[ \langle U, T_U, cPol [cInv A] \mid U \rangle \]
\[ A \mid U := \langle U, T_U, [cInv A] \mid U \rangle \]

Definition (neighbourhood)

\[ U \in Neigh A \iff \text{id}_U \in \text{Loc} \left\{ e \mid \mid e \mid \subseteq U \right\} \]

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Technische Universität Dresden

A relational localisation theory for topological algebras
How many neighbourhoods are “enough”? 

**Definition** 

Let $U \subseteq \text{Neigh}_A$. 

1. $U$ is called a cover of $A$ if $\forall U \in U: \varrho \upharpoonleft U = \sigma \upharpoonleft U \implies \varrho = \sigma$ for all $\varrho, \sigma \in c\text{Inv}_A$. 

2. $U$ is called a $c$-cover of $A$ if it is a cover of $A$ and every $U \in U$ is closed w.r.t. $T$. 

Moreover, let $E_A(U) := \bigcup \{ E_A(U) \mid U \in U \}$. 

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**Definition**

Let $\mathcal{U} \subseteq \text{Neigh } \mathbf{A}$.

1. $\mathcal{U}$ is called **cover of $\mathbf{A}$** if

   $$[\forall U \in \mathcal{U} : \varrho|_U = \sigma|_U] \Rightarrow \varrho = \sigma$$

   for all $\varrho, \sigma \in \text{clnv } \mathbf{A}$.
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   \]
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2. \( \mathcal{U} \) is called c-cover of \( \mathbf{A} \) if it is a cover of \( \mathbf{A} \) and every \( U \in \mathcal{U} \)
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How many neighbourhoods are “enough”? 

Definition
Let \( \mathcal{U} \subseteq \text{Neigh} \ A \).

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\[
\forall U \in \mathcal{U} : \varrho|_{U} = \sigma|_{U} \Rightarrow \varrho = \sigma
\]

for all \( \varrho, \sigma \in \text{clnv} \ A \).

2. \( \mathcal{U} \) is called c-cover of \( A \) if it is a cover of \( A \) and every \( U \in \mathcal{U} \)

is closed w.r.t. \( T \).

Moreover, let

\[
E_{A}(\mathcal{U}) := \bigcup \{ E_{A}(U) \mid U \in \mathcal{U} \}.
\]
Globalisation
Theorem

Let \( \mathcal{U} \subseteq \text{Neigh} \, A \). The following are equivalent:

(a) \( \mathcal{U} \) is a cover of \( A \).

(b) \( \text{id}_A \in \mathcal{L}(\mathcal{E}_A(\mathcal{U})) \).

(c) There is an index set \( \Phi \) and a map \( B : \Phi \to \{ A : \mathcal{U} \in \mathcal{U} \} \) such that \( A : \) is approximately a retract of \( \prod_{\phi \in \Phi} B : (\phi) \), i.e. there exists \( M : A : \to \prod_{\phi \in \Phi} B : (\phi) \) with \( \text{id}_A \in \mathcal{L}(\Lambda \circ M \mid \Lambda : \prod_{\phi \in \Phi} B : (\phi) \to A : \) \).
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(c) There is an index set $\Phi$ and a map $B : \Phi \to \{ \mathbf{A} \mid U \mid U \in \mathcal{U} \}$ such that $\mathbf{A}$ is approximately a retract of $\prod_{\varphi \in \Phi} B(\varphi)$, i.e. there exists $M : \mathbf{A} \to \prod_{\varphi \in \Phi} B(\varphi)$ with

$$\text{id}_A \in \text{Loc} \left\{ \Lambda \circ M \mid \Lambda : \prod_{\varphi \in \Phi} B(\varphi) \to \mathbf{A} \right\}.$$
What about the example?
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**Reminder**

Let $R = \langle R, +, -, \cdot, 0 \rangle$ be a ring.

1. $e, f \in \text{Id} R$ orthogonal $\iff e \cdot f = f \cdot e = 0$.
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Theorem (Gabriel, 1962)

Let \( R = \langle R, S, +, -, \cdot, 0, 1 \rangle \) be a compact Hausdorff topological ring, \( 0 \neq 1 \). Then there exists an orthogonal set \( E \subseteq \text{Id } R \) of primitive idempotents such that \( 1 = \sum_{e \in E} e \).
Seriously, what about the example?

\[ R = \langle R, S, +, -, \cdot, 0, 1 \rangle \] compact Hausdorff topological ring,
\[ M = \langle M, T, +, -, 0, (\lambda(r))_{r \in R} \rangle \] Hausdorff topological \( R \)-module.
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\[ R = \langle R, S, +, -, \cdot, 0, 1 \rangle \text{ compact Hausdorff topological ring,} \]
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**Lemma**
\[ \text{cNeigh } M = \{ \text{im } \lambda(e) \mid e \in \text{Id } R \}. \]
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**Theorem**
*Let* \( U \in \text{cNeigh } M, |U| > 1, \text{ and } m \in \mathbb{N}, \varrho, \sigma \in \text{clInv}^{(m)} M \) *such that* \( \varrho | U \neq \sigma | U \). TFAE:

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Let \( U \in \text{cNeigh} M, \ |U| > 1, \) and \( m \in \mathbb{N}, \ \varrho, \sigma \in \text{cInv}^{(m)} M \) such that \( \varrho \upharpoonright U \neq \sigma \upharpoonright U \). TFAE:

(i) Every c-cover of \( M|_U \) contains \( U \).
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\[ R = \langle R, S, +, -, \cdot, 0, 1 \rangle \text{ compact Hausdorff topological ring,} \]
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Let \( U \in \text{cNeigh } M, \ |U| > 1, \text{ and } m \in \mathbb{N}, \varrho, \sigma \in \text{clnv}^{(m)} M \text{ such that } \varrho|_{U} \neq \sigma|_{U}. \text{ TFAE:} \]

(i) Every c-cover of \( M|_{U} \) contains \( U \).
(ii) \( U \in \text{Min}_{\subseteq}((\text{cNeigh } M) \setminus \{ \{0\}\}). \)
Seriously, what about the example?

\[ \mathbb{R} = \langle R, S, +, -, \cdot, 0, 1 \rangle \] compact Hausdorff topological ring,
\[ \mathcal{M} = \langle M, T, +, -, 0, (\lambda(r))_{r \in R} \rangle \] Hausdorff topological \( \mathbb{R} \)-module.

**Lemma**
\[ \text{cNeigh} \mathcal{M} = \{ \text{im} \lambda(e) \mid e \in \text{Id} \mathbb{R} \}. \]

**Theorem**
Let \( U \in \text{cNeigh} \mathcal{M}, \ |U| > 1, \) and \( m \in \mathbb{N}, \varrho, \sigma \in \text{clnv}^{(m)} \mathcal{M} \) such that \( \varrho\mid_U \neq \sigma\mid_U. \) TFAE:

(i) Every c-cover of \( \mathcal{M}\mid_U \) contains \( U. \)

(ii) \( U \in \text{Min}_{\subseteq}((\text{cNeigh} \mathcal{M}) \setminus \{\{0\}\}). \)

(iii) \( U \in \text{Min}_{\subseteq}\{ V \in \text{cNeigh} \mathcal{M} \mid \varrho\mid_V \neq \sigma\mid_V \}. \)
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\[ \text{cNeigh } M = \{ \text{im } \lambda(e) \mid e \in \text{Id } R \}. \]

Theorem
Let \( U \in \text{cNeigh } M, \ |U| > 1, \) and \( m \in \mathbb{N}, \ \varrho, \sigma \in \text{cln}v^{(m)} M \) such that \( \varrho|_U \neq \sigma|_U. \) TFAE:

(i) Every c-cover of \( M|_U \) contains \( U. \)

(ii) \( U \in \text{Min}_{c}((\text{cNeigh } M) \setminus \{\{0\}\}). \)

(iii) \( U \in \text{Min}_{c}\{ V \in \text{cNeigh } M \mid \varrho|_V \neq \sigma|_V \}. \)

(iv) There exists a primitive idempotent \( e \in \text{Id } R \) such that \( U = \text{im } \lambda(e). \)
The very last slide
Thank you for your attention!!
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