Universal homogeneous constraint structures and the hom-equivalence classes of weakly oligomorphic structures

Christian Pech
Maja Pech

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Definition
A countable relational structure $A$ is called weakly oligomorphic if $\text{End}(A)$ is oligomorphic. I.e., $\text{End}(A)$ has of every arity only finitely many invariant relations on $A$.

Examples for weakly oligomorphic structures
- finite structures,
- $\omega$-categorical structures,
- retracts of weakly oligomorphic structures,
- reducts of homomorphism homogeneous structures over a finite signature
Motivation

Define $\text{CSP}(A) := \{B \mid B \text{ finite, } B \to A\}$

Theorem

If $B$ is weakly oligomorphic and $A$ is a countable structure, then the following are equivalent:

1. $A \to B$,
2. $\text{Th}^+ \exists_1(A) \subseteq \text{Th}^+ \exists_1(B)$,
3. $\text{Age}(A) \to \text{Age}(B)$,
4. $\text{CSP}(A) \subseteq \text{CSP}(B)$.

Theorem (Mašulović, MP ’11)

If $A$ is weakly oligomorphic and $B$ is countable and $B \models \text{Th}(A)$, then $B$ is weakly oligomorphic.

Corollary

Let $T$ be the first order theory of a weakly oligomorphic structure. Then all countable models of $T$ are homomorphism-equivalent.
Hom-equivalence classes

Definition
Let $A$ be a countable relational structure. Then the hom-equivalence class $E(A)$ of $A$ is the class of all countable structures $B$ such that $A \rightarrow B$ and $B \rightarrow A$.

We equip $E(A)$ with a quasiorder:
For $B, C \in E(A)$ we write $B \hookrightarrow C$ whenever there exists an embedding from $B$ into $C$.

We study the structure of $(E(A), \hookrightarrow)$,
where $A$ is a weakly oligomorphic structure.
Our first steps are to find (nice) smallest and greatest elements in $E(A)$. 
Smallest elements

Theorem
Every weakly oligomorphic relational structure $T$ is homomorphism-equivalent to a finite or $\aleph_0$-categorical substructure $C$.

Theorem (Bodirsky ’07)
Every $\aleph_0$-categorical relational structure $T$ is homomorphism-equivalent to a model-complete core $C$, which is unique up to isomorphism, and $\omega$-categorical or finite.

Corollary
For a weakly oligomorphic structure $A$ the class $\mathcal{E}(A)$ has (up to isomorphism) a unique model-complete smallest element.
Greatest elements

Theorem
Let $R$ be a countable relational signature, and let $T$ be a countable $R$-structure. Then $E(T)$ has a largest element. Moreover, if $R$ is finite and $T$ is weakly oligomorphic, then $E(T)$ has an $\omega$-categorical element.

Theorem (Saracino ’73)
Let $T$ be an $\aleph_0$-categorical theory with no finite models. Then $T$ has a model-companion $T’$. Moreover, $T’$ is $\aleph_0$-categorical, too.

Corollary
If $A$ is a weakly oligomorphic structure over a finite signature, then $E(A)$ has (up to isomorphism) a unique model-complete, $\omega$-categorical largest element.

Observation
The age of a largest element in $E(A)$ is at most $\text{CSP}(A)$.
Strict Fraïssé-classes
If \( \mathcal{C} \) is an age, then \( \overline{\mathcal{C}} := \{ A \mid A \text{ countable, } \text{Age}(A) \subseteq \mathcal{C} \} \).

Definition (Dolinka)
A Fraïssé-class \( \mathcal{C} \) of relational structures is called strict Fraïssé-class if every pair of morphisms in \( (\mathcal{C}, \hookrightarrow) \) with the same domain has a finite pushout in \( (\overline{\mathcal{C}}, \rightarrow) \).

Observation
Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.

Examples for strict Fraïssé-classes
- free amalgamation classes,
- the class of finite partial orders.

Definition
A sub-Fraïssé-class \( \mathcal{C} \) of a strict Fraïssé-class \( \mathcal{U} \) is called free in \( \mathcal{U} \) if it is closed with respect to canonical amalgams.
Universal structures

Theorem
Let $\mathcal{U}$ be a strict Fraïssé-class of relational structures, and let $\mathcal{C}$ be a Fraïssé-class that is free in $\mathcal{U}$. Let $T \in \overline{\mathcal{U}}$. Then

1. $\overline{\mathcal{C}} \cap \text{CSP}(T)$ has a universal element $U_{\mathcal{C}, T}$,

2. if the Fraïssé-limit of $\mathcal{C}$ and $T$ each have an oligomorphic automorphism group (i.e. each is finite or $\omega$-categorical), then $\overline{\mathcal{C}} \cap \text{CSP}(T)$ has a universal element $U_{\mathcal{C}, T}$ that is finite or $\omega$-categorical.

If $T \in \overline{\mathcal{C}}$, then $U_{\mathcal{C}, T}$ can be chosen as a co-retract of $T$.

Special case

$R$ is a countable relational signature, $T$ an $R$-structure, and $\mathcal{U} = \mathcal{C}$ is the class of all finite $R$-structures.
**T-colored structures**

Given

- a strict Fraïssé-class $\mathcal{U}$,
- a Fraïssé-class $\mathcal{C}$, that is free in $\mathcal{U}$, and
- $T \in \overline{\mathcal{U}}$.

**Definition**

A **$T$-colored structure in $\overline{\mathcal{C}}$** is a pair $(A, a)$ such that $A \in \overline{\mathcal{C}}$ and $a : A \to T$ is a homomorphism. The class of all such structures is denoted by $\text{Col}_C(T)$.

**Note**

A countable structure $A$ is in $\overline{\mathcal{C}} \cap \overline{\text{CSP}(T)}$ if and only if there exists $f : A \to T$ such that $(A, a)$ is a $T$-colored structure in $\overline{\mathcal{C}}$. 
Morphisms for $\mathbf{T}$-colored structures

**Strong homomorphisms**

$f : (\mathbf{A}, a) \to (\mathbf{B}, b)$ is called a strong homomorphism if $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism and $b \circ f = a$. Analogously strong embeddings and strong automorphisms are defined. $\text{sAut}(\mathbf{A}, a)$ denotes the group of strong automorphisms.

**Weak homomorphisms**

A weak homomorphism from $(\mathbf{A}, a)$ to $(\mathbf{B}, b)$ is a pair $(f, g)$ such that $f : \mathbf{A} \to \mathbf{B}$, $g \in \text{Aut}(\mathbf{T})$, $b \circ f = g \circ a$. If $f$ is an embedding (an automorphism), then $(f, g)$ is called a weak embedding (a weak automorphism). Composition is component-wise. $\text{wAut}(\mathbf{A}, a)$ denotes the group of weak automorphisms.

$c\text{Aut}(\mathbf{A}, a) := \{ f \in \text{Aut}(\mathbf{A}) \mid \exists g \in \text{Aut}(\mathbf{T}) : (f, g) \in \text{wAut}(\mathbf{A}, a)\}.$

**Remark**

- We have $f : (\mathbf{A}, a) \to (\mathbf{B}, b)$ iff $(f, 1_\mathbf{T}) : (\mathbf{A}, a) \to (\mathbf{B}, b)$.
- If $a$ is surjective, then $c\text{Aut}(\mathbf{A}, a) \cong \text{wAut}(\mathbf{A}, a)$. 
Universal homogeneous $T$-colored structures

**Theorem**

There exists $(U, u) \in \text{Col}_C(T)$ such that

1. for every $(A, a) \in \text{Col}_C(T)$ there exists an embedding $\iota : (A, a) \hookrightarrow (U, u)$ (universality),

2. for every finite $(A, a) \in \text{Col}_C(T)$, and for all $\iota_1, \iota_2 : (A, a) \hookrightarrow (U, u)$ there exists $f \in \text{sAut}(U, u)$ such that $f \circ \iota_1 = \iota_2$ (homogeneity).

Moreover, all countable universal homogeneous $T$-colored structures are mutually isomorphic.

**Remark**

- If $F$-$\text{Lim}(C)$ is finite or $\omega$-categorical, and if $T$ is finite, then $\text{sAut}(U, u)$ is oligomorphic.
- If $T \in \overline{C}$, then $T$ is a retract of $U$. 
**w-homogeneity**

**Definition**

$(U, u) \in \text{Col}_C(T)$ is called **w-homogeneous** if for every finite $(A, a) \in \text{Col}_C(T)$, and for $(f_1, g_2), (f_2, g_2) : (A, a) \hookrightarrow (U, u)$ there exists $(f, g) \in w\text{Aut}(U, u)$ such that $(f, g) \circ (f_1, g_1) = (f_2, g_2)$.

**Proposition**

Let $(U, u) \in \text{Col}_C(T)$ be universal and homogeneous. Then $(U, u)$ is w-homogeneous, too.

**Remark**

- If $F\text{-Lim}(C)$ is finite or $\omega$-categorical, and if $T$ is finite or $\omega$-categorical, too, then $c\text{Aut}(U, u)$ is oligomorphic.
Universal homogeneous objects in categories

Definition
We call a category $\mathcal{C}$ a $\lambda$-amalgamation category if
1. all morphisms of $\mathcal{C}$ are monomorphisms,
2. $\mathcal{C}$ is $\lambda$-algebroidal,
3. $\mathcal{C}_{<\lambda}$ has the joint embedding property,
4. $\mathcal{C}_{<\lambda}$ has the amalgamation property.

Theorem (Droste, Göbel ’92)
Let $\lambda$ be a regular cardinal, and let $\mathcal{C}$ be a $\lambda$-algebroidal category in which all morphisms are monomorphisms. Then there exists a $\mathcal{C}$-universal, $\mathcal{C}_{<\lambda}$-homogeneous object in $\mathcal{C}$ if and only if $\mathcal{C}$ is a $\lambda$-amalgamation category. Moreover, any two $\mathcal{C}$-universal, $\mathcal{C}_{<\lambda}$-homogeneous objects in $\mathcal{C}$ are isomorphic.
Amalgamation pairs

Definition
A pair of categories \((\mathcal{A}, \hat{\mathcal{A}})\) is called a \(\lambda\)-amalgamation pair if
1. \(\mathcal{A} \leq \hat{\mathcal{A}}\) is isomorphism closed,
2. all morphisms of \(\mathcal{A}\) are monomorphisms,
3. \(\mathcal{A}\) is \(\lambda\)-algebroidal,
4. \(\mathcal{A}_{<\lambda}\) has the free joint embedding property in \(\hat{\mathcal{A}}\), and
5. \(\mathcal{A}_{<\lambda}\) has the free amalgamation property in \(\hat{\mathcal{A}}\).

Remark
\(\lambda\)-amalgamation pairs are a category-theoretic version of the idea of free amalgamation classes and of strict amalgamation classes.
Theorem

Let \((\widehat{\mathcal{A}}, \mathcal{A})\) be a \(\lambda\)-amalgamation pair, \(\mathcal{B}\) be a \(\lambda\)-amalgamation category, and let \(\mathcal{C}\) be a category. Let \(\widehat{F} : \widehat{\mathcal{A}} \to \mathcal{C}\), \(G : \mathcal{B} \to \mathcal{C}\) and let \(F\) be the restriction of \(\widehat{F}\) to \(\mathcal{A}\). Further suppose that

1. \(\widehat{F}\) preserves weak coproducts and weak pushouts in \(\mathcal{A}_{<\lambda}\),
2. \(F\) and \(G\) are \(\lambda\)-continuous,
3. \(F\) preserves \(\lambda\)-smallness,
4. \(G\) preserves monomorphisms,
5. for every \(A \in \mathcal{A}_{<\lambda}\) and for every \(B \in \mathcal{B}_{<\lambda}\) there are at most \(\lambda\) morphisms in \(\mathcal{C}(FA \to GB)\).

Then \((F \downarrow G)\) has a \((F \downarrow G)\)-universal, \((F \downarrow G)_{<\lambda}\)-homogeneous object. Moreover, up to isomorphism there is just one such object in \((F \downarrow G)\).
Definition
A Fraïssé-class $\mathcal{C}$ has the Hrushovski property if for every $A \in \mathcal{C}$ there exists a $B \in \mathcal{C}$ such that $A \leq B$ and such that every isomorphism between substructures of $A$ extends to an automorphism of $B$.

Definition
Let $G \leq S_\omega$. Then $G$ is said to have the small index property if every subgroup of index less than $2^{\aleph_0}$ contains the stabilizer of a finite tuple (i.e. subgroups of small index are open in the topology of pointwise convergence on $G$).

Remark
- The Hrushovski-property of a free amalgamation class $\mathcal{C}$ implies the small index property of the automorphism group of $\text{F-Lim}(\mathcal{C})$.
- The Hrushovski-property can straight-forwardly be defined for Fraïssé-classes of finite constraint structures.
Link-structures
A finite $R$-structure $A$ is called a link-structure, if either $|A| = 1$ or there exist $a_1, \ldots, a_n \in A$ such that $A = \{a_1, \ldots, a_n\}$ and for some $\varrho \in R^{(n)}$ we have $(a_1, \ldots, a_n) \in \varrho_A$.

Link-type
If $\mathcal{L}$ is a set of link-structures, then we say that a structure $A$ has link type $\mathcal{L}$ if every substructure of $A$ that is a link structure, is isomorphic to some structure from $\mathcal{L}$.

Free monotone amalgamation classes
A free amalgamation class is called monotone if it is a CSP, too.

Definition
Let $\mathcal{C}$ be a free monotone amalgamation class, $\mathcal{L}$ be a set of link-structures. By $\mathcal{C}_\mathcal{L}$ we denote the class of all structures from $\mathcal{C}$ whose link-type is $\mathcal{L}$.

Remark
$\mathcal{C}_\mathcal{L}$ is a free amalgamation class, too.
Definition
A finite structure is called **sparse** if it has only finitely many non-empty basic relations. A relational structure is called sparse if all finite substructures are sparse.

Theorem
Let $R$ be any relational signature, let $\mathcal{C}$ be a free, monotone amalgamation class, and let $\mathcal{L}$ be a countable set of sparse link-structures. Let $T$ be any countable $R$-structure. Then $\text{Col}_{\mathcal{C}_L}(T)$ has the Hrushovski property. If $(U, u)$ is a universal homogeneous $T$-colored structure in $\mathcal{C}_L$, then $\text{sAut}(U, u)$ has the small index property.

Remark
- The proof uses an adapted version of a criterion for the (SIP) due to Herwig (which in turn generalizes Hrushovski’s ideas from graphs to relational structures).
- If $\text{sAut}(U, u)$ is oligomorphic, then it has uncountable cofinality and the Bergman-property. (Kechris, Rosendal)