How many Higher Commutator Operations can we Define on a Given Congruence Lattice of a Mal’cev Algebra?

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Motivation

PROBLEM: Describe all clones of operations on a finite set that contain all polynomial operations.
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**HOPE:** Every such a clone is determined by the set of all unary polynomials and all higher commutators.
Higher Commutators

In general, introduced by A. Bulatov in 2001:

\[[\bullet, \ldots, \bullet]\]

as an \(n\)-ary operation, for each \(n \geq 2\), on the lattice of congruences that satisfies the certain centralizing condition.
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**Definition for expanded groups:**

**Theorem.** (E. Aichinger and \( \sim \), published 2010) If \( A_1, \ldots, A_n \in \text{Id } V, V = \langle V, +, F \rangle \) then \([A_1, \ldots, A_n]\) is an ideal generated by the set

\[ \{ p(a_1, \ldots, a_n) | a_i \in A_i, 1 \leq i \leq n, p \in \text{Pol}_n V \} \]

such that \( p(x_1, \ldots, x_n) = 0 \) whenever \( \exists i \) such that \( x_i = 0 \).
Properties of Higher Commutators in Mal’cev Algebras

**Theorem.** (E. Aichinger and Ė., published 2010)

Let $A$ be a Mal’cev algebra, $n \geq 2$, $I \neq \emptyset$, and

$\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \text{Con} A$

(HC1) $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_{i=1}^{n} \alpha_i$

(HC2) $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n \Rightarrow [\alpha_1, \ldots, \alpha_n] \leq [\beta_1, \ldots, \beta_n]$

(HC3) $[\alpha_1, \ldots, \alpha_n] \leq [\alpha_2, \ldots, \alpha_n]$

(HC4) $[\alpha_1, \ldots, \alpha_n] = [\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}]$, for all permutations $\pi$ on $\{1, \ldots, n\}$

(HC7) $\bigvee_{i \in I} [\alpha_1, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] = [\alpha_1, \ldots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_k]$, for all $j \in \{1, \ldots, k\}$

(HC8) $[\alpha_1, \ldots, \alpha_j, [\alpha_{j+1}, \ldots, \alpha_k]] \leq [\alpha_1, \ldots, \alpha_k]$, for all $j \in \{1, \ldots, k-2\}$.  

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The Sequence of Higher Commutators

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This sequence satisfies the properties

(HC1), (HC2), (HC3), (HC4), (HC7), (HC8).
Sequences of Operations

Let $L$ be a lattice. We call the set $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ a sequence of operations on the lattice $L$ if it satisfies the following properties:

\[(HC3) \quad f_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) \leq f_n(\alpha_2, \ldots, \alpha_{n+1})\]
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(HC8) $f_k(\alpha_1, \ldots, \alpha_{k-1}, f_{n-k+1}(\alpha_k, \ldots, \alpha_n)) \leq f_n(\alpha_1, \ldots, \alpha_n)$ where $k \in \{2, \ldots, n-1\}$ and $n \neq 2$,

for all $n \geq 2$ and for all $\alpha_1, \ldots, \alpha_n \in L$. 
Sequences With Additional Properties

We say that sequences of operations $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ on a lattice $L$ satisfies

(HC1) if $f_n(\alpha_1, \ldots, \alpha_n) \leq \bigwedge_{i=1}^{n} \alpha_i$,

(HC2) if $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n \Rightarrow f_n(\alpha_1, \ldots, \alpha_n) \leq f_n(\beta_1, \ldots, \beta_n)$,

(HC7) if $\bigvee_{i \in I} f_n(\alpha_1, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_n) = f_n(\alpha_1, \ldots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_n)$,

for all $n \geq 2$, $I \neq \emptyset$, $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in L$, $\{\rho_i \mid i \in I\} \subseteq L$ and $j \in \{1, \ldots, n\}$. 
Our Goal

Given: A congruence lattice $L$ of a finite Mal’cev algebra.
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Question 1: How many sequences of operations that satisfy (HC1), (HC2) and (HC7) can we define on $L$?
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**Theorem.** (E. Aichinger, 2009) Let $L$ be a finite lattice. The number of sequences of functions $\langle f_1, f_2, \ldots \rangle$ on $L$ that satisfy (HC3) and (HC4) is at most countable.
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Answer to Question 1: There are at most countably many sequences of operations on $L$ that satisfy (HC1), (HC2) and (HC7).
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Answer to Question 1: There are at most countably many sequences of operations on $L$ that satisfy (HC1), (HC2) and (HC7).

Question 2: Are there always infinitely many such sequences?
The Splitting Property

**Definition.** Let $L$ be a lattice with the least element 0 and the largest element 1. We say that $L$ splits if

$$(\exists \delta, \epsilon \in L)(\delta < 1 \land \epsilon > 0 \land (\forall \alpha \in L)(\alpha \leq \delta \lor \alpha \geq \epsilon)).$$

Then, we call $(\delta, \epsilon)$ a splitting pair.
The Splitting Property

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**Examples:**

The diamond does not split. The other three lattices on the picture split.
The Splitting Property

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![Diagram of lattices](image-url)
The Strong Splitting Property

**Definition.** We say that $L$ **strongly splits** if it splits and

$$(\exists \delta, \varepsilon \in L) (\delta < 1 \land \varepsilon > 0 \land (\forall \alpha \in L) (\alpha \leq \delta \lor \alpha \geq \varepsilon)$$

$$(\land (\varepsilon \leq \delta \text{ or } \varepsilon \text{ is not an atom or } \delta \text{ is not a coatom})).$$

**Examples:** The two element lattice and $M_2$ do not split strongly, but the third lattice does. We denote it by $L_s$. 

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*Proof:* Let $\alpha, \beta, \gamma \notin \{0, 1\}$. Using $(HC1)$ and $(HC7)$ we obtain

$$f(1, 1, 1) = f(\beta \lor \gamma, \alpha \lor \gamma, \alpha \lor \beta) = 0.$$
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**Proposition.** There are finitely many sequences of operations on the diamond that satisfy properties (HC1), (HC2) and (HC7).
The Two Element Lattice

**Proposition.** Let $L$ be a lattice with the least element $0$ and the largest element $1$ and let $\theta$ be an atom of $L$. If $\langle f_k | k \geq 2 \rangle$ is a sequence of operations on $L$ that satisfies (HC1) then $f_k(\theta, \ldots, \theta) = \theta$ for all $k \geq 2$ or $f_k(\theta, \ldots, \theta) = 0$ for all $k \geq 2$. 
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**Proof:** Let us suppose that there exists an $n \in \mathbb{N}$ such that $f_n(\theta, \ldots, \theta) \neq 0$. Hence, $f_n(\theta, \ldots, \theta) = \theta$, because $\theta$ is an atom. We will prove that

$$f_k(\theta, \ldots, \theta) = \theta \text{ for all } k \geq 2. \quad \Box$$
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**Proposition.** There are only two sequences of operations that satisfy (HC1), (HC2) and (HC7) on the two element lattice.
How Do We Proceed?

**Proposition.** Let $\delta, \epsilon \in M_2 \setminus \{0, 1\}$. Then, for each $n \geq 2$ we have

$$f(\delta, \ldots, \delta) = g(\delta, \ldots, \delta) \land f(\epsilon, \ldots, \epsilon) = g(\epsilon, \ldots, \epsilon) \implies f = g,$$

for all $n$-ary functions $f$ and $g$ on $M_2$ that satisfy (HC1) and (HC7).
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**Proof:** If $(x_1, \ldots, x_n) \in \{\delta, \epsilon\}^n$ and $(x_1, \ldots, x_n) \not\in \{(\delta, \ldots, \delta), (\epsilon, \ldots, \epsilon)\}$ then, using (HC1) we have $f(x_1, \ldots, x_n) = 0.$
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We know $\delta \lor \epsilon = 1$. Using (HC7) we obtain the statement. 

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**Proposition.** There are finitely many sequences of operations on the lattice \( M_2 \) that satisfy properties (HC1), (HC2) and (HC7).
The Example of a Strongly Splitting Lattice

**Proposition.** Let \( n \geq 2 \) and \((\delta, \epsilon)\) the splitting pair of the lattice \( L_s \). If we define \( f_n : L^n_s \to L_s \) such that

\[
f_n(\alpha_1, \ldots, \alpha_n) := \begin{cases} 
0 & , (\exists i) \alpha_i \leq \delta \\
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then $f_n$ satisfies (HC1), (HC2) and (HC7).

**Proposition.** The following sequences
\[
  \langle f_2, 0, 0, \ldots \rangle, \langle f_2, f_3, 0, \ldots \rangle, \ldots
\]
are infinitely many sequences of operations on the lattice $L_s$ that satisfy (HC1), (HC2) and (HC7).
The Answer

**Theorem.** Let $A$ be a Mal’cev algebra with a finite congruence lattice $L$. There are infinitely many sequences of operations on the lattice $L$ that satisfy properties (HC1), (HC2) and (HC7) if and only if $L$ splits strongly.