New Maximal Subsemigroups of the Semigroup of all Transformations on a countable set

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• 2002 L. Heindorf: The maximal clones on countable sets that include all permutations

• 2005 M. Pinsker: Clones on uncountable sets that include all permutations

• April 2011 J. East, D. Mitchell, Y. Péresse: Maximal subsemigroups of the semigroup of all mappings on an infinite set containing all permutations

• September 2011 J. East, D. Mitchell, Y. Péresse: Maximal subsemigroups of the semigroup of all mappings on an infinite set.
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Classification of the maximal subsemigroups of the semigroup of all mappings on an infinite set $\Omega$ that contains one of the following subgroups of the symmetric group on $\Omega$: 

- setwise stabilizer of a non-empty finite subset of $\Omega$
- the stabilizer of a finite partition of $\Omega$
- the stabilizer of an ultrafilter on $\Omega$. 
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...containing a particular semigroup U

- Let \( \Omega \) be countable
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$\Omega^\Omega$ semigroup of all mappings on the set $\Omega$
...containing a particular semigroup $U$

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- $U \subset \Omega^\Omega$
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$W \leq \Omega^\Omega$, where each $\alpha \in U$ is a generator modulo $W$
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- $W \leq S \leq \Omega^\Omega$
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**Problem**

*Characterization of all maximal subsemigroups of $S$*
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**Definition**

For $M \subseteq \mathcal{P}(\Omega \Omega)$, let $J(M)$ be the set of all $A \subseteq \bigcup M$ with
\[
\forall m \in M (A \cap m \neq \emptyset) \land \forall a \in A \exists m \in M (A \cap m = \{a\})
\]
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**Definition**

For $M \subseteq \mathcal{P}(\Omega^\Omega)$, let $J(M)$ be the set of all $A \subseteq \bigcup M$ with

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**Definition**

For $U \subseteq \Omega^\Omega$ and $W \subseteq \Omega^\Omega$, we put

$Gen(U) := \{A \subseteq \Omega^\Omega \mid A \text{ is finite and } \langle A \rangle \cap U \neq \emptyset\}$ and

$\mathcal{H}(U, W) := \{A \subseteq \Omega^\Omega \setminus W \mid A \in J(Gen(U))\}$
Main theorem

Theorem

Let $W \leq S \leq \Omega^\Omega$ and $U \subset \Omega^\Omega$ such that each $\alpha \in U$ is a generator modulo $W$. Then the following statements are equivalent:

(i) $S$ is maximal.

(ii) There is a set $H \in \mathcal{H}(U, W)$ with $S = \Omega^\Omega \setminus H$. 

Jörg Koppitz (Institute)
The maximal subsemigroups containing the symmetric group

- $\text{Inj}(\Omega)$ the set of all injective but not surjective mappings on $\Omega$
The maximal subsemigroups containing the symmetric group

- \( Inj(\Omega) \) the set of all injective but not surjective mappings on \( \Omega \)
- \( Sur(\Omega) \) the set of all surjective but not injective mappings on \( \Omega \)
The maximal subsemigroups containing the symmetric group

- $\text{Inj}(\Omega)$ the set of all injective but not surjective mappings on $\Omega$
- $\text{Sur}(\Omega)$ the set of all surjective but not injective mappings on $\Omega$
- $\text{C}_p(\Omega) := \{ \alpha \in \Omega^\Omega \mid \text{rank}(\alpha) = \aleph_0 \text{ and } k(\alpha) = \aleph_0 \}$

\textbf{Theorem (L. Heindorf 2002)}
Let $S$ contain the symmetric group. $S$ is maximal if $S = \Omega^\Omega$ for some $H \in \{ \text{Inj}(\Omega), \text{Sur}(\Omega), \text{C}_p(\Omega), \text{IF}(\Omega), \text{FI}(\Omega) \}$.
The maximal subsemigroups containing the symmetric group

- \( \text{Inj}(\Omega) \) the set of all injective but not surjective mappings on \( \Omega \)
- \( \text{Sur}(\Omega) \) the set of all surjective but not injective mappings on \( \Omega \)
- \( C_p(\Omega) := \{ \alpha \in \Omega^\Omega \mid \text{rank}(\alpha) = \aleph_0 \text{ and } k(\alpha) = \aleph_0 \} \)
- \( k(\alpha) := |\{ x \in \text{im} \alpha \mid |x \alpha^{-1}| = \aleph_0 \} | \) (infinite contraction index of \( \alpha \))
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- \( d(\alpha) := |\Omega \setminus \text{im}\alpha| \) (defect of \( \alpha \))
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- $c(\alpha) := \sum_{x \in \text{im}\alpha} (|x\alpha^{-1}| - 1)$ (collapse of $\alpha$)
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- $k(\alpha) := |\{ x \in \text{im}\alpha \mid |x\alpha^{-1}| = \mathbb{N}_0 \}|$ (infinite contraction index of $\alpha$)
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- $c(\alpha) := \sum_{x \in \text{im}\alpha} (|x\alpha^{-1}| - 1)$ (collapse of $\alpha$)
- $IF(\Omega) := \{ \alpha \in \Omega^\Omega \mid \text{rank}(\alpha) = \mathbb{N}_0, c(\alpha) = \mathbb{N}_0, \text{ and } d(\alpha) < \mathbb{N}_0 \}$
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**Theorem**

(L. Heindorf 2002)

Let $S \leq \Omega^\Omega$ containing the symmetric group. $S$ is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \{\text{Inj}(\Omega), \text{Sur}(\Omega), C_p(\Omega), \text{IF}(\Omega), \text{FI}(\Omega)\}$
The maximal subsemigroups containing $\text{Inj}(\Omega)$ or $\text{Sur}(\Omega)$
Sur(X) and Inj(X)

- The maximal subsemigroups containing $\text{Inj}(\Omega)$ or $\text{Sur}(\Omega)$

Theorem

Let $S \leq \Omega^\Omega$ containing $\text{Sur}(\Omega)$. $S$ is maximal iff $S = \Omega^\Omega \setminus \text{Inj}(\Omega)$ or $S = \Omega^\Omega \setminus \text{FI}(\Omega)$. 
The maximal subsemigroups containing $\text{Inj}(\Omega)$ or $\text{Sur}(\Omega)$

**Theorem**

Let $S \leq \Omega^\Omega$ containing $\text{Sur}(\Omega)$. $S$ is maximal iff $S = \Omega^\Omega \setminus \text{Inj}(\Omega)$ or $S = \Omega^\Omega \setminus \text{FI}(\Omega)$.

**Theorem**

Let $S \leq \Omega^\Omega$ containing $\text{Inj}(\Omega)$. $S$ is maximal iff $S = \Omega^\Omega \setminus \text{Sur}(\Omega)$ or $S = \Omega^\Omega \setminus \text{IF}(\Omega)$ or $S = \Omega^\Omega \setminus \text{C}_p(\Omega)$. 
maximal subsemigroups containing $FI(\Omega)$ (using main theorem)
maximal subsemigroups containing $\text{FI} (\Omega)$ (using main theorem)

Lemma

$\text{FI}(\Omega)$ is a subsemigroup of $\Omega^\Omega$.
maximal subsemigroups containing $FI(\Omega)$ (using main theorem)

**Lemma**

$FI(\Omega)$ is a subsemigroup of $\Omega^\Omega$.

**Lemma**

Each $\alpha \in C_p(\Omega) \cap Sur(\Omega)$ is a generator modulo $FI(\Omega)$. 
maximal subsemigroups containing $\text{FI}(\Omega)$ (using main theorem)

**Lemma**

$\text{FI}(\Omega)$ is a subsemigroup of $\Omega^\Omega$.

**Lemma**

Each $\alpha \in C_p(\Omega) \cap \text{Sur}(\Omega)$ is a generator modulo $\text{FI}(\Omega)$.

**Theorem**

Let $S \leq \Omega^\Omega$ containing $\text{FI}(\Omega)$. $S$ is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \mathcal{H}(C_p(\Omega) \cap \text{Sur}(\Omega), \text{FI}(\Omega))$. 
maximal subsemigroups containing $IF(\Omega)$ (using main theorem)
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**Lemma**

$IF(\Omega)$ is a subsemigroup of $\Omega^\Omega$. 
maximal subsemigroups containing $IF(\Omega)$ (using main theorem)

**Lemma**

$IF(\Omega)$ is a subsemigroup of $\Omega^\Omega$.

**Lemma**

Each $\alpha \in FI(\Omega) \cap \text{Inj}(\Omega)$ is a generator modulo $IF(\Omega)$.
maximal subsemigroups containing $IF(\Omega)$ (using main theorem)

Lemma

$IF(\Omega)$ is a subsemigroup of $\Omega^\Omega$.

Lemma

Each $\alpha \in FL(\Omega) \cap Inj(\Omega)$ is a generator modulo $IF(\Omega)$.

Theorem

Let $S \leq \Omega^\Omega$ containing $IF(\Omega)$. $S$ is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \mathcal{H}(Inj(\Omega) \cap FL(\Omega), IF(\Omega))$. 
maximal subsemigroups containing $C_p(\Omega)$ (using main theorem)
- maximal subsemigroups containing $C_p(\Omega)$ (using main theorem)

**Lemma**

Each $\alpha \in Fl(\Omega) \cap Inj(\Omega)$ is a generator modulo $\langle C_p(\Omega) \rangle$. 
maximal subsemigroups containing $C_p(\Omega)$ (using main theorem)

**Lemma**

*Each* $\alpha \in FL(\Omega) \cap Inj(\Omega)$ *is a generator modulo* $\langle C_p(\Omega) \rangle$.

**Theorem**

*Let* $S \leq \Omega^\Omega$ *containing* $IF(\Omega)$. *$S$ is maximal iff* $S = \Omega^\Omega \setminus H$ *for some* $H \in H(Inj(\Omega) \cap FI(\Omega), C_p(\Omega))$. 
