The asymptotic number of ways to intersect two composition series

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14th March 2012

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Define $\mathrm{CSL}_G(\vec{H}, \vec{K}) := (\{H_i \cap K_j : i, j \in \{0, \ldots, n\}\}; \subseteq)$

*http://www.math.u-szeged.hu/∼czedli/

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Prop: Assume $k = p_1 \dots p_n$ and L is lower semimodular, meet-generated by two chains, and length$(L) = n$. Then the cyclic C_k group of order k has $\vec{H},\vec{K}$ with $L \cong \mathsf{CSL}_{C_k}(\vec{H},\vec{K})$.

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Describes what we want to count.

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By duality, it suffices to count **slim** (= join-generated by two chains) **semimodular** lattices of length n, asymptotically.

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2. Count permutations (Cz-O-U).

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Part I: description by permutations.

Thm (Cz-Sch): Slim semimodular (planar) **diagrams** ↔ permutations.

Need: a pair of reciprocal bijections.

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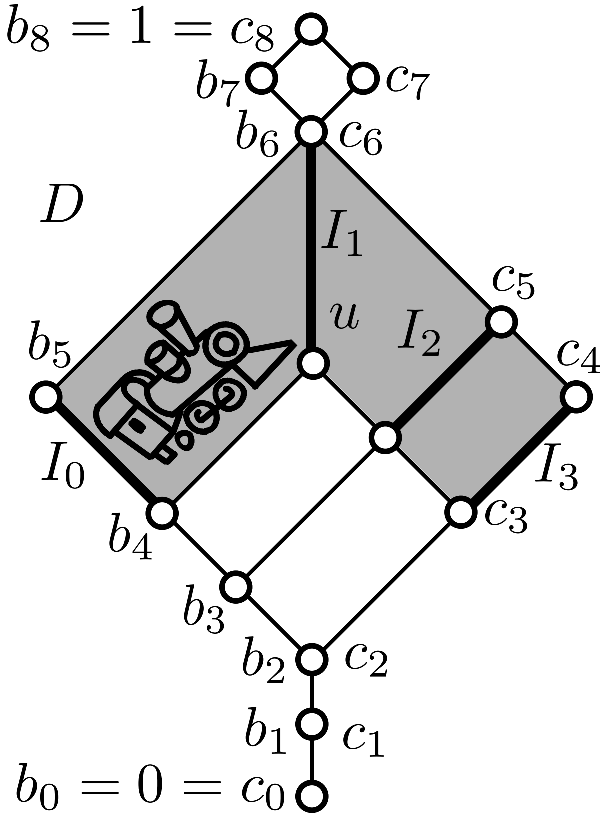
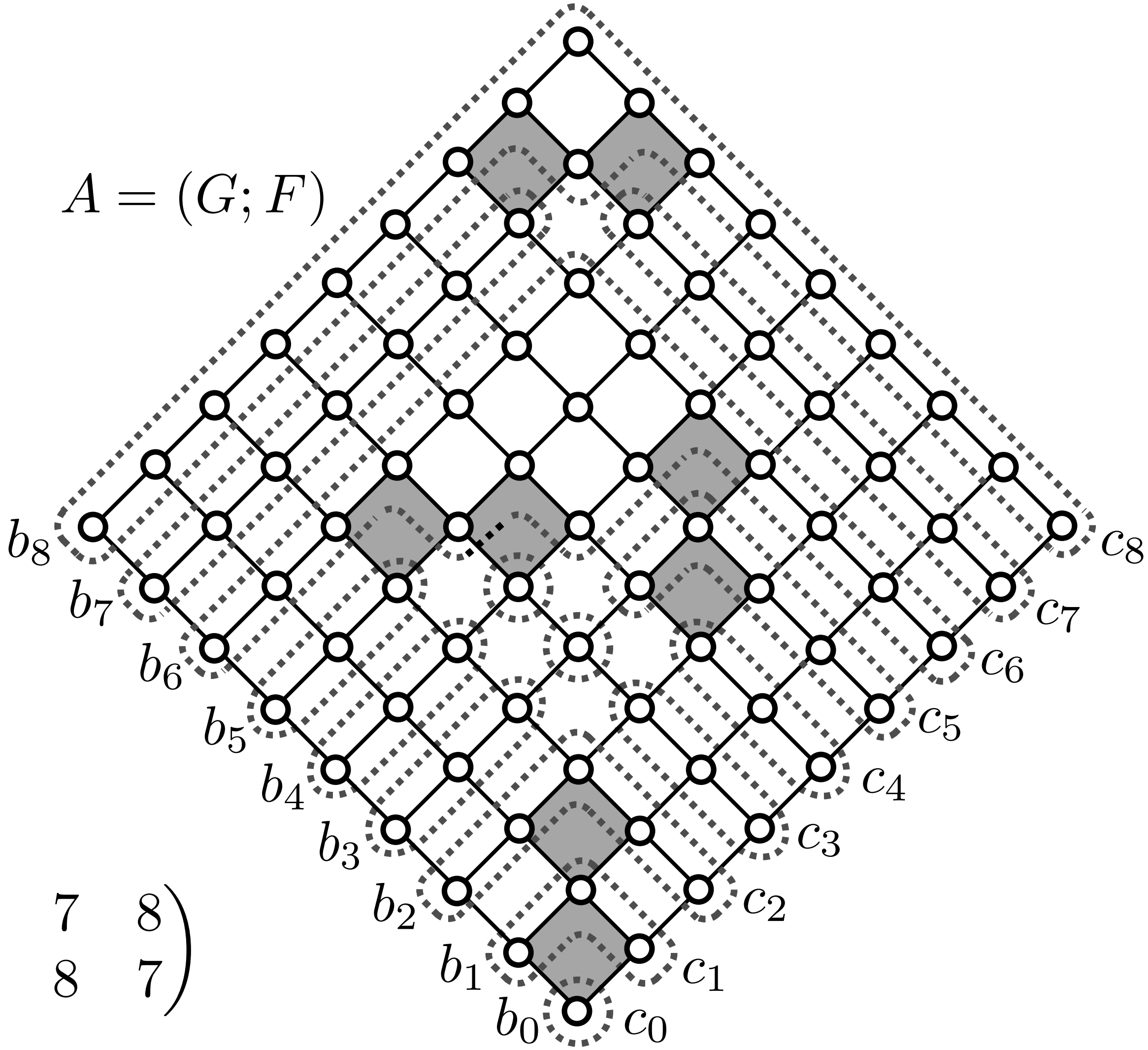
$D \mapsto \pi$ by a **locomotive**. $\pi \mapsto D$ by quotient join-semilattice.





$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 4 & 3 & 8 & 7 \end{pmatrix}$$

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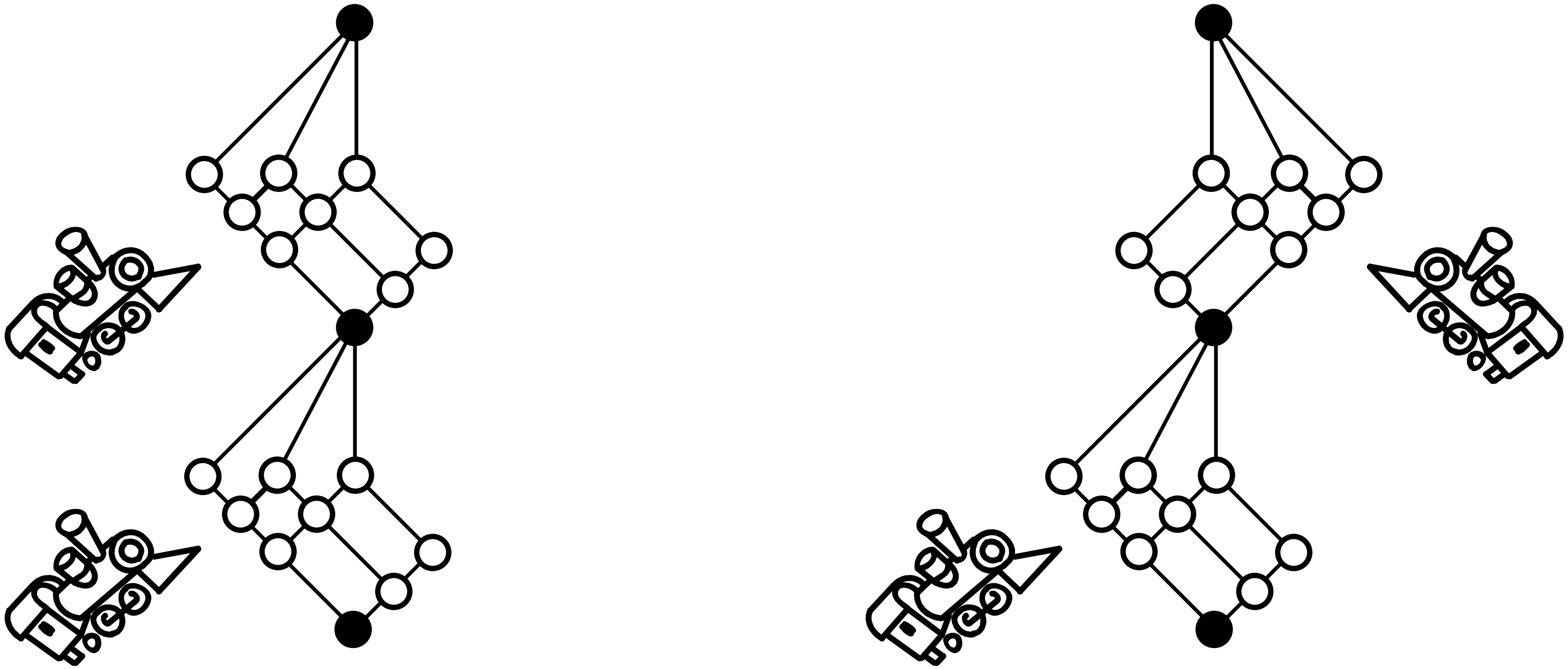
Reflecting a segment $\iff$ inverting the restriction of π



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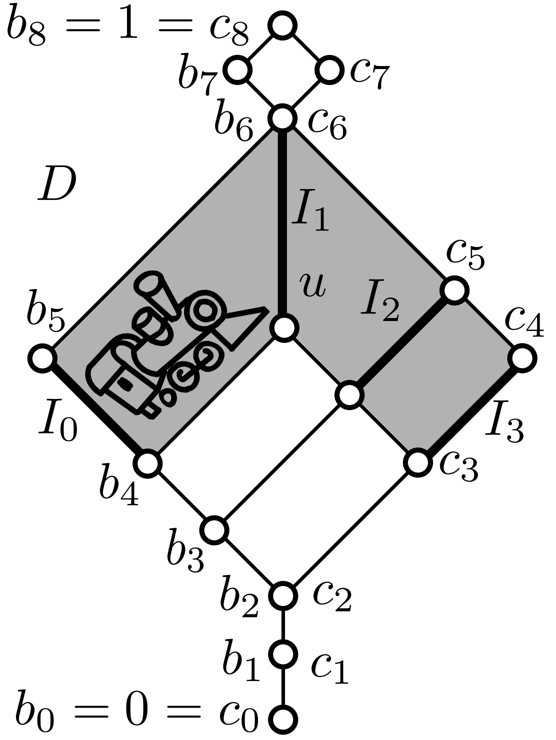

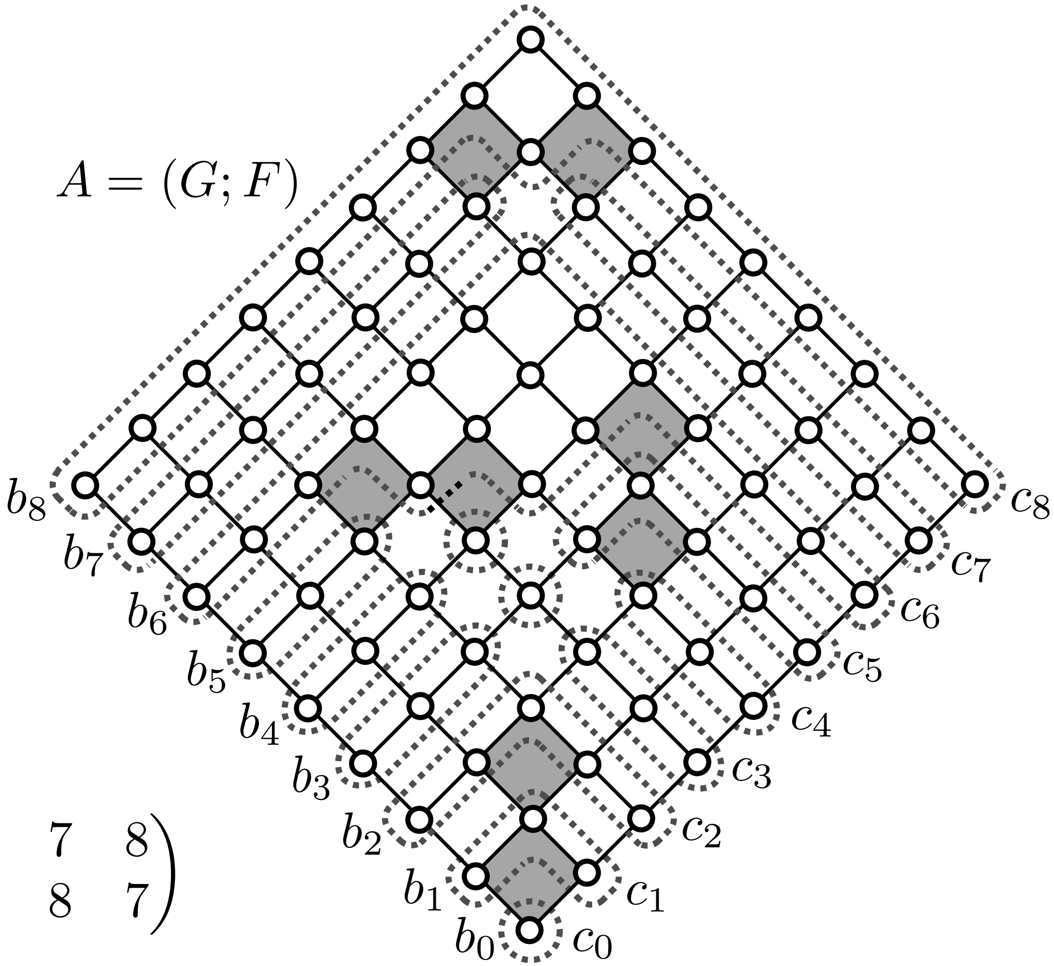

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The segments of π are $\{1\}$, $\{2\}$, $\{3, 4, 5, 6\}$, $\{7, 8\}$.

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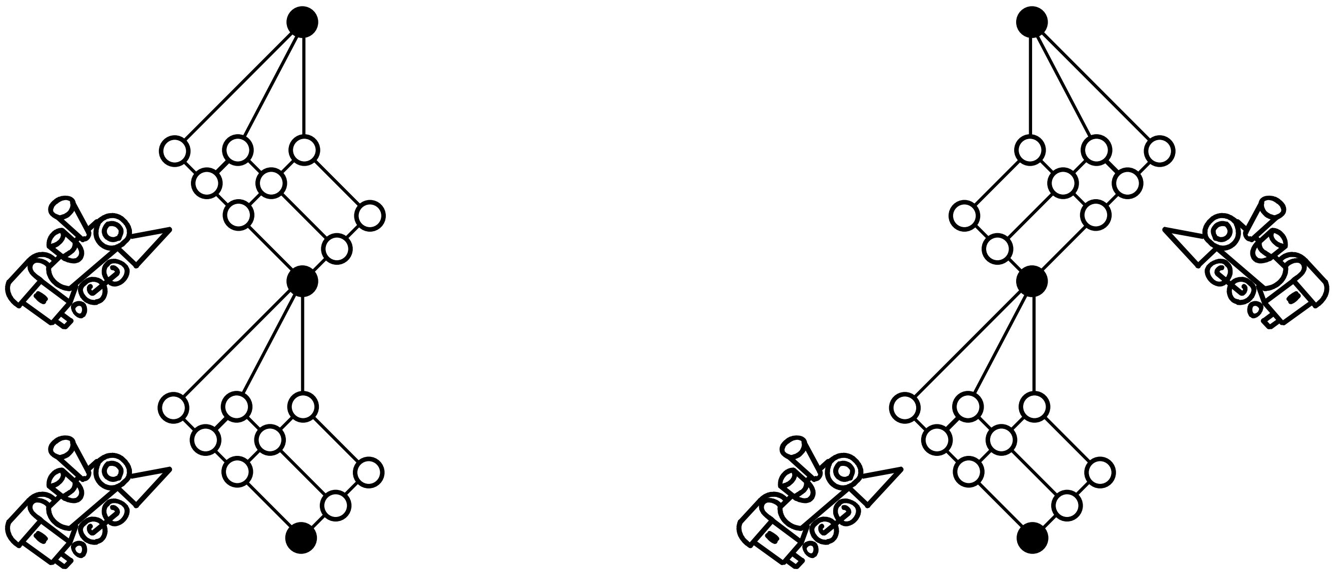

Lemma: $L(\pi) \cong L(\tau)$ iff π and τ are "sectionally inverse or equal", denoted by $\pi \sim \tau$.

It suffices to determine $|S_n/\sim|$, asymptotically.

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$$\frac{|A_0(n)|}{n!} = \sum_{j=1}^{\lfloor n/4\rfloor} \frac{1}{(n-2j)!\cdot 2^j\cdot j!} + \sum_{j=\lfloor n/4\rfloor+1}^{\lfloor n/2\rfloor} \frac{1}{(n-2j)!\cdot 2^j\cdot j!} = \sum' + \sum''.$$

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In $\sum'$, each denominator is at least $(n-2\lfloor n/4 \rfloor)! \geq \lfloor n/2 \rfloor!$, and there are fewer than n summands. Hence $\sum' \leq n\cdot(\lfloor n/2 \rfloor!)^{-1} \to 0$.

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Large segment: consists of at least 3 elements.

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Since $\lambda \in S_{p(\pi)}$, there are at most $p(\pi)!$ many such λ. (In fact, there are much fewer.) Similarly, there are at most $(n-p(\pi))!$ many ρ.

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Taking the well-known fact

$$\binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \geq \binom{n}{\lceil n/2 \rceil + 1} \geq \cdots \geq \binom{n}{n-3}$$

at $\leq^*$ into account and counting the permutations according to their pivot elements, we obtain:

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*http://www.math.u-szeged.hu/∼czedli/