Lattice polynomial functions and their use in qualitative decision making
AAA83

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March 2012
Main Problem: Model preference!
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Model: $R$ on $X_1 \times \cdots \times X_n$ is represented by $f: X_1 \times \cdots \times X_n \rightarrow X$:

$$xRy \iff f(x) \leq f(y)$$

Limitation: The role of local preferences is not explicit!
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Aggregation: $x_1, \ldots, x_n \quad \longrightarrow \quad y = A(x_1, \ldots, x_n)$
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Let \( X \) be a scale (bounded chain).

An aggregation function on \( X \) is a mapping \( A: X^n \rightarrow X \) such that:

1. **A is order-preserving:** for every \( x, y \in X^n \)
   \[
   x \leq y \implies A(x) \leq A(y)
   \]

2. **A preserves the boundaries:**
   \[
   \inf_{x \in X^n} A(x) = \inf X \quad \text{and} \quad \sup_{x \in X^n} A(x) = \sup X.
   \]

Traditionally: \( X \) is a real interval \( I \subseteq \mathbb{R} \), e.g., \( I = [0, 1] \).
Numerical representation of relations: \( f : X_1 \times \cdots \times X_n \to I \subseteq \mathbb{R} : \)

\[ x R y \iff f(x) \leq f(y) \]

**DM:*** Preference on criteria \( i \) is represented by a local utility function

\[ \varphi_i : X_i \to I. \]

Preference on \( X_1 \times \cdots \times X_n \) is represented by an overall utility function:

\[ F(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \]

where \( A : I^n \to I \) is an aggregation function.
Examples of aggregation functions:

1. **Arithmetic means:** For \( x \in \mathbb{I}^n \),
   \[
   AM(x) := \frac{1}{n} \sum_{1 \leq i \leq n} x_i
   \]

2. **Weighted arithmetic means:** For \( x \in \mathbb{I}^n \) and \( \sum w_i = 1 \),
   \[
   WAM(x) := \sum_{1 \leq i \leq n} w_i x_i
   \]

3. **Choquet integrals:** For \( x \in \mathbb{I}^n \),
   \[
   C(x) := \sum_{I \subseteq \{1, \ldots, n\}} a_I \cdot \bigwedge_{i \in I} x_i
   \]
Qualitative decision making **QDM**

**In the qualitative approach:**

The underlying sets $X_1, \ldots, X_n$ and $X$ are finite chains (ordinal scales), e.g., $X = \{\text{very bad, bad, satisfactory, good, very good}\}$

**QDM:** Preference relation on $X_i$ is represented by

$$\varphi_i : X_i \rightarrow X.$$ 

Preference relation on $X_1 \times \cdots \times X_n$ is represented by

$$F(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $A : X^n \rightarrow X$ is an aggregation function.
Let $X$ be a chain with least and greatest elements 0 and 1, respectively.

A capacity is a mapping $v : 2^{[n]} \rightarrow X$, $[n] = \{1, \ldots, n\}$, such that

1. $v(I) \leq v(J)$ whenever $I \subseteq J$,

2. $v(\emptyset) = 0$ and $v([n]) = 1$. 

 Capacities
Order simplexes of $X^n$

Let $\sigma$ be a permutation on $[n] = \{1, \ldots, n\}$ ($\sigma \in S_n$)

\[ X^n_\sigma = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in X^n : x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \right\} \]

**Example:** $X = [0, 1]$ and $n = 2$

\[ x_1 \leq x_2 \]

\[ x_1 \geq x_2 \]

$2! = 2$ permutations (2 simplexes)
The (discrete) Sugeno integral on $X$ w.r.t. $\nu$ is defined by

$$S_{\nu}(x) := \bigvee_{i \in [n]} \nu(\{\sigma(i), \ldots, \sigma(n)\}) \land x_{\sigma(i)}$$

for every $x \in X_{\sigma}^n = \{(x_1, \ldots, x_n) \in X^n : x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\}$.

**Example**

If $x_3 \leq x_1 \leq x_2$, then $x_{\sigma(1)} = x_3$, $x_{\sigma(2)} = x_1$, $x_{\sigma(3)} = x_2$, and

$$S_{\nu}(x_1, x_2, x_3) = (\nu(\{1, 2, 3\}) \land x_3) \lor (\nu(\{1, 2\}) \land x_1) \lor (\nu(\{2\}) \land x_2)$$

$$= 1$$
Qualitative decision making QDM

Setting:

1. $n$ criteria on finite chains $X_1, \ldots, X_n$
2. scores in a common finite chain $X$ by local utility functions $\varphi_i : X_i \rightarrow X$

We will assume that each $\varphi_i$ is order-preserving.

3. Preference relation on $X_1 \times \cdots \times X_n$ is represented by

$$F(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $A : X^n \rightarrow X$ is a Sugeno integral. We shall refer to these overall utility functions as Sugeno utility functions.
Outline

1 Preliminaries: Sugeno integrals as lattice polynomial functions.

2 Characterizations of lattice polynomial functions.
Outline

1. Preliminaries: Sugeno integrals as lattice polynomial functions.

2. Characterizations of lattice polynomial functions.

3. Generalization of polynomial functions: Sugeno utility functions.

4. Sugeno utility functions: characterizations and factorizations.
Outline

1. Preliminaries: Sugeno integrals as lattice polynomial functions.
2. Characterizations of lattice polynomial functions.
3. Generalization of polynomial functions: Sugeno utility functions.
4. Sugeno utility functions: characterizations and factorizations.
5. Axiomatic approach to qualitative decision-making QDM.
6. Further research directions and open problems.
Let $X$ be a distributive (finite) lattice with

1. operations $\land$ and $\lor$,
2. least and greatest elements 0 and 1, respectively.
A (lattice) polynomial function (on \( X \)) is any map \( p : X^n \to X, \ n \geq 1 \), obtainable by finitely many applications of the rules:

1. The projections \( x \mapsto x_i, \ i \in [n] \), and the constant functions \( x \mapsto c, \ c \in X \), are polynomial functions.

2. If \( f : X^n \to X \) and \( g : X^n \to X \) are polynomial functions, then \( f \land g \) and \( f \lor g \) are polynomial functions.

Example

\[
\text{median}(x_1, x_2, x_3) = (x_1 \land x_2) \lor (x_2 \land x_3) \lor (x_3 \land x_1)
\]
A function $f : X^n \rightarrow X$ has a disjunctive normal form (DNF) if

$$f(x) = \bigvee_{I \subseteq [n]} (a_I \land \bigwedge_{i \in I} x_i).$$
A function \( f : X^n \rightarrow X \) has a **disjunctive normal form (DNF)** if

\[
f(x) = \bigvee_{I \subseteq [n]} \left( \bigwedge_{i \in I} a_i \wedge \bigwedge_{i \in I} x_i \right).
\]

**Proposition (Goodstein’67)**

A function \( p : X^n \rightarrow X \) is a polynomial function iff it has the DNF:

\[
p(x) = \bigvee_{I \subseteq [n]} \left( p(1_I) \wedge \bigwedge_{i \in I} x_i \right)
\]

where \( 1_I \) denotes the “characteristic tuple” of \( I \subseteq [n] \).
The Sugeno integral on a chain $X$ w.r.t. $v: 2^n \to X$ is defined by

$$S_v(x) := \bigvee_{i \in [n]} v(\{\sigma(i), \ldots, \sigma(n)\}) \land x_{\sigma(i)}$$

for every $x \in X^*_\sigma = \{(x_1, \ldots, x_n) \in X^n : x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\}$. 
Sugeno integrals as lattice polynomial functions

The Sugeno integral on a chain $X$ w.r.t. $v : 2^{[n]} \rightarrow X$ is defined by

$$S_v(x) := \bigvee_{i \in [n]} v(\{\sigma(i), \ldots, \sigma(n)\}) \land x_{\sigma(i)}$$

for every $x \in X_{\sigma}^n = \{(x_1, \ldots, x_n) \in X^n : x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\}$.

**Theorem (Marichal)**

A function $q : X^n \rightarrow X$ is the Sugeno integral $S_v$ iff

$$q(x) = \bigvee_{l \subseteq [n]} (v(l) \land \bigwedge_{i \in l} x_i).$$

Since, $q(1_l) = v(l)$, and $v(\emptyset) = 0$ and $v([n]) = 1$, Sugeno integrals coincide with **idempotent** polynomial functions: $q(x, \ldots, x) = x$. 
General properties of polynomial functions

Fact

Every polynomial function (in part., Sugeno integral) is order-preserving.

However...

The function $f(0) = f(a) = 0$ and $f(1) = 1$ is order-preserving on \{0, a, 1\}, but it is not a polynomial function, hence not a Sugeno integral!
Median decomposability (Marichal)

For $c \in X$ and $i \in [n]$, set $x_i^c = (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n)$. A function $f : X^n \to X$ is median decomposable if for each $i \in [n]$

$$f(x) = \text{median} \left( f(x_0^i), x_i, f(x_1^i) \right), \quad \text{for every } x \in X^n.$$
Median decomposability (Marichal)

For $c \in X$ and $i \in [n]$, set $\mathbf{x}_i^c = (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n)$.

A function $f : X^n \to X$ is median decomposable if for each $i \in [n]$

$$f(\mathbf{x}) = \text{median} \left( f(\mathbf{x}_i^0), x_i, f(\mathbf{x}_i^1) \right), \quad \text{for every } \mathbf{x} \in X^n.$$
Characterization of polynomial functions

Fact
Every median decomposable function is order-preserving.
Characterization of polynomial functions

**Fact**
Every median decomposable function is order-preserving.

**Theorem (Marichal)**
A function $p: X^n \rightarrow X$ is
1. a polynomial function iff it is median decomposable.
2. a Sugeno integral iff it is idempotent and median decomposable.
General criterion (C. & Marichal)

Let $C$ be a class of functions such that

(i) the unary members of $C$ are polynomial functions;

(ii) any $g : X \to X$ obtained from $f : X^n \to X \in C$ by fixing $n - 1$ arguments is in $C$.

Then $C$ is a class of polynomial functions.
Let $X := X_1 \times \cdots \times X_n$, where each $X_i$ is a finite distributive lattice.
Extensions: pseudo-polynomial functions

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each $X_i$ is a finite distributive lattice.

**Definition**

We say that $f : \mathbf{X} \to X$ is a pseudo-polynomial function if

$$f(x) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n)),$$

where $p: X^n \to X$ is polynomial function and each $\varphi_i: X_i \to X$ satisfies

$$\varphi_i(0) \leq \varphi_i(x_i) \leq \varphi_i(1).$$  \hspace{1cm} (BC)

**Fact:** We can always choose $p$ to be a Sugeno integral!
A function \( f : X \rightarrow X \) is a **Sugeno utility function** if

\[
    f(x) = q(\varphi_1(x_1), \ldots, \varphi_n(x_n)),
\]

where \( q \) is a Sugeno integral and each \( \varphi_i : X_i \rightarrow X \) is order-preserving.
A function \( f : X \rightarrow X \) is a Sugeno utility function if

\[
f (x) = q (\varphi_1 (x_1), \ldots, \varphi_n (x_n)),
\]

where \( q \) is a Sugeno integral and each \( \varphi_i : X_i \rightarrow X \) is order-preserving.

**Proposition (C. & Waldhauser)**

Order-preserving pseudo-polynomial functions are Sugeno utility functions.
Consider $f: X \rightarrow X$.

**Problem 1:** Determine whether $f$ is pseudo-polynomial function.

**Problem 2:** Find all possible factorizations $f = p(\varphi_1, \ldots, \varphi_n)$. 

Remark: Problems 1 and 2 were solved (C. & Marichal) when $X_1 = \cdots = X_n$ and $f = p(\varphi(x_1), \ldots, \varphi(x_n))$. Such model is pertaining to QDM under uncertainty.
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**Problem 1:** Determine whether $f$ is pseudo-polynomial function.

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$$f = p(\varphi(x_1), \ldots, \varphi(x_n)).$$

Such model is pertaining to **QDM under uncertainty**.
We say that \( f : X \rightarrow X \) is pseudo-median decomposable if for each \( i \in [n] \) there exists \( \varphi_i : X_i \rightarrow X \) such that

\[
f(x) = \text{median} \left( f(x^0_i), \varphi_i(x_i), f(x^1_i) \right), \quad \text{for all } x \in X.
\]
We say that $f : X \to X$ is **pseudo-median decomposable** if for each $i \in [n]$ there exists $\varphi_i : X_i \to X$ such that

$$f(x) = \text{median} \left( f(x_i^0), \varphi_i(x_i), f(x_i^1) \right), \quad \text{for all } x \in X.$$
If $f$ is pseudo-median decomposable w.r.t. $\varphi_i$, then $f = p_f(\varphi_1, \ldots, \varphi_n)$.

where $p_f(x) = \bigvee_{I \subseteq [n]} (f(\hat{1}_I) \wedge \bigwedge_{i \in I} x_i)$. 
Characterizations of pseudo-polynomial functions (I)

**Proposition (C. & Waldhauser)**

If \( f \) is pseudo-median decomposable w.r.t. \( \varphi_i \), then \( f = \rho_f(\varphi_1, \ldots, \varphi_n) \)

where \( \rho_f(x) = \bigvee_{I \subseteq [n]} (f(1_I) \land \bigwedge_{i \in I} x_i) \).

**Theorem (C. & Waldhauser)**

\( f \) is a pseudo-polynomial function \( \text{iff} \) it is pseudo-median decomposable.
Embedding a distributive lattice $X$ into a power-set $Y$
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Closure and interior operators on $Y$

closure operator: $\text{cl}(b) = \bigwedge_{a \in X} a$

interior operator: $\text{int}(b) = \bigvee_{a \in X} a$
Closure and interior operators on $Y$

closure operator: $\text{cl}(b) = \bigwedge_{a \in X} a$

interior operator: $\text{int}(b) = \bigvee_{a \in X} a$

$\text{cl}(\overline{D}) = \text{cl}(\overline{N}) = \text{cl}(\overline{G}) = V$

$\text{int}(\overline{D}) = N, \text{ int}(\overline{N}) = D, \text{ int}(\overline{G}) = B$
Towards necessary conditions...

Given \( f : X \to X \) and \( i \in [n] \), define functions \( \Phi_i^-, \Phi_i^+ : X_i \to X \) by

\[
\Phi_i^- (a_i) := \bigvee_{x_i = a_i} \cl (f(x) \land f(x_i^0)),
\]
\[
\Phi_i^+ (a_i) := \bigwedge_{x_i = a_i} \int (f(x) \lor f(x_i^1)).
\]
Towards necessary conditions...

Given \( f : X \to X \) and \( i \in [n] \), define functions \( \Phi_i^- , \Phi_i^+ : X_i \to X \) by

\[
\Phi_i^- (a_i) := \bigvee_{x_i = a_i} \text{cl}( f(x) \land f(x_i^0)) , \\
\Phi_i^+ (a_i) := \bigwedge_{x_i = a_i} \text{int}( f(x) \lor f(x_i^1)).
\]

Proposition (C. & Waldhauser)

If \( f : X \to X \) is a pseudo-polynomial function, then

\[
f = p_f (\varphi_1, \ldots, \varphi_n) , \text{ for } \varphi_i \in \{ \Phi_i^-, \Phi_i^+ \}.
\]
Characterization of pseudo-polynomial functions

Fact

If $f$ is a pseudo-polynomial function, then it satisfies

$$f(x_i^0) \leq f(x) \leq f(x_i^1).$$

($\text{BC}_n$)
Characterization of pseudo-polynomial functions

Fact

If $f$ is a pseudo-polynomial function, then it satisfies

$$f(x_i^0) \leq f(x) \leq f(x_i^1).$$

(BC$_n$)

Theorem (C. & Waldhauser)

The function $f$ is a pseudo-polynomial function iff

1. $f$ satisfies (BC$_n$)
2. for every $i \in [n]$, $\Phi_i^- \leq \Phi_i^+$. 
When $X$ is a finite chain

**Theorem (C. & Waldhauser):** For a finite chain $X$...

$f : X \rightarrow X$ is pseudo-polynomial **iff** it satisfies $(BC_n)$ and

$$f(x^0_i) < f(x^a_i) \text{ and } f(y^a_i) < f(y^1_i) \implies f(x^a_i) \leq f(y^a_i)$$
Finding the local utility functions

Theorem (C. & Waldhauser)

A function \( \varphi_i : X_i \rightarrow X \) satisfying (BC) appears in a factorization of \( f \) iff

\[
\Phi_i^- \leq \varphi_i \leq \Phi_i^+.
\]
Finding all polynomial functions

Let \( f : X \to X \) and \( \varphi_i : X_i \to X \) be given as before.

We define the polynomial functions \( p^- , p^+ : Y^n \to X \) by

\[
p^- (y) := \bigvee_{I \subseteq [n]} (c^-_I \wedge \bigwedge_{i \in I} x_i) \quad \text{with} \quad c^-_I := \text{cl} \left( f(\hat{1}_I) \wedge \bigwedge_{i \notin I} \varphi_i(0) \right),
\]

\[
p^+ (y) := \bigvee_{I \subseteq [n]} (c^+_I \wedge \bigwedge_{i \in I} x_i) \quad \text{with} \quad c^+_I := \text{int} \left( f(\hat{1}_I) \vee \bigvee_{i \in I} \varphi_i(1) \right).
\]
Finding all polynomial functions

Let \( f : X \rightarrow X \) and \( \varphi_i : X_i \rightarrow X \) be given as before.

We define the polynomial functions \( p^-, p^+ : Y^n \rightarrow X \) by

\[
p^-(y) := \bigvee_{I \subseteq [n]} (c_I^- \land \bigwedge_{i \in I} x_i) \quad \text{with} \quad c_I^- := \text{cl}(f(\hat{1}_I) \land \bigwedge_{i \notin I} \varphi_i(0)),
\]

\[
p^+(y) := \bigvee_{I \subseteq [n]} (c_I^+ \land \bigwedge_{i \in I} x_i) \quad \text{with} \quad c_I^+ := \text{int}(f(\hat{1}_I) \lor \bigvee_{i \in I} \varphi_i(1)).
\]

**Theorem (C. & Waldhauser)**

For a polynomial function \( p(y) = \bigvee_{I \subseteq [n]} (c_I \land \bigwedge_{i \in I} x_i) \) we have

\( f = p(\varphi_1, \ldots, \varphi_n) \) if and only if \( c_I^- \leq c_I \leq c_I^+ \) holds for all \( I \subseteq [n] \).
Main Problems

1. Model preference relations.
2. Axiomatize the chosen model.

Question: What is a preference relation?
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1. Model preference relations.

2. Axiomatize the chosen model.

Question: What is a preference relation?
Let $X := X_1 \times \cdots \times X_n$, where each $X_i$ is a finite chain.
Preference relations

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each $X_i$ is a finite chain.

A **weak order** on $\mathbf{X}$ is a relation $\preceq \subseteq \mathbf{X}^2$ that is:

1. **reflexive:** $\forall x \in \mathbf{X} : x \preceq x$,
2. **transitive:** $\forall x, y, z \in \mathbf{X} : x \preceq y, y \preceq z \implies x \preceq z$, and
3. **complete:** $\forall x, y \in \mathbf{X} : x \preceq y$ or $y \preceq x$.

Note: Weak orders are not necessarily antisymmetric: $\forall x, y \in \mathbf{X} : x \preceq y, y \preceq x \implies x = y$ (AS)
Let \( X := X_1 \times \cdots \times X_n \), where each \( X_i \) is a finite chain.

A \textbf{weak order} on \( X \) is a relation \( \preceq \subseteq X^2 \) that is:

1. \( \text{reflexive:} \) \( \forall x \in X : x \preceq x \),
2. \( \text{transitive:} \) \( \forall x, y, z \in X : x \preceq y, y \preceq z \implies x \preceq z \), and
3. \( \text{complete:} \) \( \forall x, y \in X : x \preceq y \) or \( y \preceq x \).

\textbf{Note:} Weak orders are not necessarily antisymmetric:

\[ \forall x, y \in X : x \preceq y, y \preceq x \implies x = y \quad \text{(AS)} \]
The **indifference relation** $\sim$ associated with $\preceq$ is defined by:

$$y \sim x \text{ iff } x \preceq y \text{ and } y \preceq x.$$ 

**Note that...**

1. $\sim$ is an equivalence relation.
2. $\leq := \preceq / \sim$ satisfies (AS) and $\mathbf{X} / \sim$ is a (finite) chain.
A preference relation on $X$ is a weak order $\preceq$ that satisfies

**Pareto condition:** $\forall x, y \in X : \forall i \in [n], x_i \preceq_i y_i \implies x \preceq y$. 

Consequence: Preference relations are exactly those representable by order-preserving functions.
A preference relation on $X$ is a weak order $\preceq$ that satisfies

**Pareto condition:** $\forall x, y \in X : \forall i \in [n], x_i \preceq_i y_i \implies x \preceq y$.

**Fact**

The rank function $r : X \rightarrow X/ \sim$ of $\preceq$ is order-preserving and:

$$x \preceq y \iff r(x) \leq r(y).$$
Preference relations

A preference relation on $X$ is a weak order $\preceq$ that satisfies the Pareto condition:

**Pareto condition:** $\forall x, y \in X : \forall i \in [n], x_i \preceq_i y_i \implies x \preceq y$.

**Fact**

The rank function $r : X \rightarrow X/\sim$ of $\preceq$ is order-preserving and:

$$x \preceq y \iff r(x) \leq r(y).$$

**Consequence:**

Preference relations are exactly those representable by order-preserving functions.
Model: Preference relations are represented by Sugeno utility functions.
Axiomatic approach to QDM

Model: Preference relations are represented by Sugeno utility functions.

Theorem (C. & Dubois & Waldhauser)

A relation $\preceq$ on $X$ is representable by a Sugeno utility function iff

1. $\preceq$ is a preference relation
2. $\preceq$ satisfies: $\forall x, y \in X: x_i^0 \prec x_i^a$ and $y_i^a \prec y_i^1 \implies x_i^a \preceq y_i^a$. 
Proof

Theorem: For a finite chain $X$...

$f : X \rightarrow X$ is a Sugeno utility function iff it is order-preserving and

$$f(x^0_i) < f(x^{a_i}_i) \text{ and } f(y^{a_i}_i) < f(y^1_i) \implies f(x^{a_i}_i) \leq f(y^{a_i}_i) \quad (*)$$
Proof

**Theorem:** For a finite chain $X$...

$f : X \rightarrow X$ is a Sugeno utility function *iff* it is order-preserving and

$$f(x_i^0) < f(x_i^{a_i}) \text{ and } f(y_i^{a_i}) < f(y_i^1) \implies f(x_i^{a_i}) \leq f(y_i^{a_i}) \quad (\star)$$

If $\preceq$ is a preference relation satisfying:

$$\forall x, y \in X : x_i^0 \prec x_i^{a_i} \text{ and } y_i^{a_i} \prec y_i^1 \implies x_i^{a_i} \preceq y_i^{a_i},$$

then $r$ is a Sugeno utility function representing $\preceq$. 

Conversely...  

**Theorem:** For a finite chain $X$...  

$f: X \rightarrow X$ is a Sugeno utility function \textit{iff} it is order-preserving and  

$$f (x^0_i) < f (x^a_i) \text{ and } f (y^a_i) < f (y^1_i) \implies f (x^a_i) \leq f (y^a_i) \quad (*)$$

**Conversely,** suppose $\preceq$ is represented by a Sugeno utility function $f$.  

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**Theorem:** For a finite chain $X$...

$f: X \to X$ is a Sugeno utility function *iff* it is order-preserving and

\[ f(x^0_i) < f(x^{a_i}_i) \quad \text{and} \quad f(y^a_i) < f(y^1_i) \implies f(x^{a_i}_i) \leq f(y^a_i) \quad (\ast) \]

Conversely, suppose $\preceq$ is represented by a Sugeno utility function $f$.

Then we may assume that $f$ is surjective.

Hence $r = \alpha \circ f$ for some order-isomorphism $\alpha$. 
Conversely...

**Theorem:** For a finite chain $X$...

$f : X \rightarrow X$ is a Sugeno utility function if and only if it is order-preserving and

$$f(x_i^0) < f(x_i^a) \text{ and } f(y_i^a) < f(y_i^1) \implies f(x_i^a) \leq f(y_i^a) \quad (\ast)$$

Conversely, suppose $\preceq$ is represented by a Sugeno utility function $f$.

Then we may assume that $f$ is surjective.

Hence $r = \alpha \circ f$ for some order-isomorphism $\alpha$.

Since $f$ satisfies $(\ast)$, $r$ satisfies $(\ast)$ and thus

$$\forall x, y \in X : x_i^0 \prec x_i^a \text{ and } y_i^a \prec y_i^1 \implies x_i^a \preceq y_i^a. \quad \square$$
Remarks:

**QDM under uncertainty:** Single universe $X_0 = X_1 = X_2 = \cdots = X_n$ and a single utility function $\varphi: X_0 \to X$ for each $i \in [n]$.

1. Computational approach: Chateauneuf & Grabisch & Labreuche & Rico

2. Axiomatic treatment: Dubois & Fargier & Prade & Sabbadin
Further problems and directions of research:

1. Properties for aggregation (functional equations):
   **Examples:** associativity, commutation, scale invariance...

2. Aggregation on specific scales:
   **Examples:** ordinal, interval, bipolar scales...

3. Interpolation problems:
   **Applications in AI:** learning functions and preferences...

4. Fusion of (qualitative) information.

5. Construction methods.

6. ...
Thank you for your attention!