Introduction
“A dozen easy problems”

- Presentation by McNulty at the 2010 AMS Special Session in St. Paul
- Contains 6 new and 6 revived problems in Universal Algebra
- Important footnote:

“Well, easily formulated …”
Question 6
Automatic Algebra Problems

● “Which finite automatic algebras are dualizable?”
● “Is the automatic algebra drawn below dualizable?”
What are Automatic Algebras?

- Start with a (partial) automaton without initial and terminal states
- Transition labels are colour coded (for this presentation)
Automatic Algebras - Elements

- **States:**
- **Transition types (letters):**
- **Default state:**

![Diagram showing states and transition types with labeled states 0, 1, 2, 3 and letters b and r.]
Automatic Algebras
One binary operation \(\circ\)

- If letter \(a\) induces a transition from state \(n\) to state \(m\), let \(n \circ a = m\)

- \(1 \circ b = 2\)
- \(2 \circ y = 2\)
- \(2 \circ r = 3\)
Automatic Algebras

One binary operation \( \circ \)

- All other products evaluate to the default state \( 0 \)
- \( 0 \circ a = 0 \)
- \( b \circ b = 0 \)
- \( 2 \circ 3 = 0 \)
- \( 2 \circ b = 0 \)
What are Dualities?

- Well ....

- Ross Willard once introduced them like this ...

Definition

A finite algebra $\mathbf{A}$ is dualizable if

- there exists an “alter ego” $\mathbf{M}$ . . .
- . . . partial operations . . . relations . . . discrete topology . . .
- . . . ISP and $\text{IS}_c P^+$ . . .
- . . . contravariant hom-functors . . .
- . . . dual adjunction $(\mathcal{D}, \mathcal{E}, e, \varepsilon)$ . . .
- AARRRGHH!!! STOP THE INSANITY!!
Why dualities for automatic algebras?

- Lyndon’s example of an algebra that is not finitely based is an automatic algebra.
- Automatic algebras have been examined for finite basedness by Boozer and have given interesting examples.
- Dualizability is another finiteness condition.
- Its relationship with finite basedness is not well-known.
Why ? (2)

- This algebra is non-finitely based, but “just fails” to be inherently non-finitely based
Results

- If an automatic algebra is dualizable then every letter acts as whiskery cycles

- If a letter acts otherwise, it is inherently non-dualizable (i.e. all its superalgebras are non-dualizable)
Whiskery Cycles
Inherently non-dualizable
Classification

- An algebra with exactly one letter is dualizable if and only if the letter acts as whiskery cycles.
Results

- If each letter acts as a total constant function, then the algebra is dualizable
Results

- If each edge is a loop, then the algebra is dualizable
Classification

● All automatic algebras with exactly one non-zero state are dualizable

● An algebra with exactly two non-zero states is dualizable if and only if
  o each letter acts as whiskery cycles
  o the algebra is pseudo-commutative, i.e.
    satisfies \((x \circ y) \circ z \circ w = (x \circ z) \circ y \circ w\)
Classification

- All dualizable automatic algebras with 2 states and at least 2 letters are (essentially)
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Unfortunately …

- whiskery cycles + pseudo commutativity does not work if there are more states

Non-dualizable (but not inherently so!)
Commutative permutations

● Each letter acts as a permutation of the non-zero states and all such actions commute

● So we can identify the letters with members of an Abelian group of permutations
Result

● For an automaton as above, if the letter actions form a coset of the corresponding Abelian permutation group, then the algebra is dualizable

● In particular,
  o if there is just one permutation, the algebra is dualizable
  o if the letter actions form an Abelian group the algebra is dualizable
Unfortunately …

- the converse is false

**dualizable**
Negative Results

- For connected automata, “certain” non-singleton cosets must be present among the letter actions in dualizable algebras.

- In particular, if the algebra has at least two distinctly acting letters, but less than needed to form the smallest non-trivial coset, the algebra is not dualizable.
How difficult is the (easily formulated) problem?

- We are able to construct a chain of superalgebras that are alternatively dualizable and non-dualizable

- Such a chain has previously only been constructed in the class of unary algebras
Thank you!